18.091
Lecture 1
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1. What is a dynamical system?
    1.1. Any system that changes over time
        a) Continual: Orbits
        b) Discrete: Compound Interest

2. Iteration: A process that is repeated over and over again.
    a) The output is sent back into the function as the new input
    b) *Notation:* The n-th iteration is written as $F^n(x)$

    2.2. Example 1: For compounding interest at a rate of 10% per year:

    A_0 = \$100
    A_1 = \$110
    A_2 = \$121
    A_3 = ...

    a) Can be modeled by an iteration I(x) = 1.1x

    A_0 = \$100
    A_1 = I(100) = 1.1*100 = \$110
    A_2 = I(121) = I(I(100)) = \$121
    $A_3 = I^3(A)$

    b) Can be evaluated in general using $A_n = (1.1)^n A$

    2.3. Example 2: Finding Square Roots:
    a) Need an algorithm. To find $\sqrt{n}$:
        1) Make a guess x
        2) Average x and n/x
        3) Use the result as your new guess
        4) Repeat until guess is good enough

    b) Proof of Method:

    Given x, n > 0, we have two cases:

Case 1:	Case 2:
$\sqrt{n} < x$	$\sqrt{n} > x$
$n < x\sqrt{n}$	$n > x\sqrt{n}$
$n/x < \sqrt{n}$	$n/x > \sqrt{n}$

    In either case, $\sqrt{n}$ is between n and n/G, so by averaging, we narrow the range in which $\sqrt{n}$ can lie.

c) For n=10 and an initial guess x=1

Iteration	***Guess***	***Guess^2***	***Average***
1	1	1	5.5
2	5.5	30.25	3.65909
3	3.65909	13.3889	3.30493
4	3.30493	10.9226	3.16536
5	3.16536	10.0195	

2.4. Changing from continuous to discrete:

a) Continually-compounded Interest:

$A(t) = A_0e^{kt}$ [t= time in years] ----> $A_n = A_0e^{kn}$ [n= # of years]

- Changed from continuous to discrete with an iteration $I(x) = (e^k)x$

b) Planetary Orbits

Draw an imaginary plane in space and look at the points where the orbit intersects the plane

c) Gains some simplicity at the expense of some information

- Discrete, instead of continuous
- No information about behavior in between iterations

3. Orbits:

3.1. Informally - The outputs of an iteration listed in the order that they are achieved

3.2. Formally – Given $x_0 \in \mathbf{R}$, the orbit of x_0 under F is the sequence of points x_0, x_1, x_2 such that $x_n = F^n(x_0)$

3.3. Useful things we can say about orbits:

a) Limit as $n \to \infty$

- $S(x) \to \sqrt{n}$

b) Are there any patterns?

4. Types of Orbits:

4.1. Fixed Points: *Definition:* $F(x_0) = x_0$, for some x_0

- If $a=\sqrt{n}$, $S(a)=a$

4.2. Periodic Orbits / Cycles: *Definition:* $F^k(x_0) = x_0$, for some k, x_0

- If $F(x) = 5 - x$

  $F(5) = 0$ $F(0) = 5$
- Called a 2-cycle

b) Finding a k-cycle

- Solve the equation $F^k(x) = x$
- If F is a quadratic function, this has degree 2^k. In general, that is impossible to solve exactly.

c) Note that if F has a k-cycle, then it has cycles of length nk, for all integers n

- $F^{3k}(x_0) = F^k(F^{2k}(x_0)) = F^{2k}(x_0) = . . . = x_0$
- Prime Period: n=1

4.3. Eventually Fixed: *Definition:* $F^k(x_0) = x^*$, for all k sufficiently large

a) $F(x) = x^2 - 1$, $x_0=(\sqrt{5} + 1)/2$

$(\sqrt{5} + 1)/2, 0, 0, 0, ...$

4.4. Eventually Periodic: *Definition:* $F^m(x_0) = F^n(x_0)$, for some m,n greater than 1

a) $F(x) = x^2 - 1$, $x_0=\sqrt{(\sqrt{2} + 1)}$

$\sqrt{(\sqrt{2}+1)}, \sqrt{2}, 1, 0, -1, 0, -1, 0, ...$

5. Computers

5.1. Uses:

a) Visualizing orbits

b) Approximating values:
- solving F(x)=x
- finding square roots

5.2. Shortfalls:

a) Rounding Errors!
- Table on page 23
- If x is close enough to zero, the computer will round to zero, and the orbit will become fixed, instead of remaining chaotic like it should.