18.091 Lecture 6
Symbolic Dynamics

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1 Introduction

This is a very interesting topic in the study of chaotic dynamical systems, which includes "bits and pieces" from areas such as Analysis, Topology and Group Theory. The main purpose of today's lecture is to gain a deeper understanding of the chaotic behaviour of the family of functions Q_c when $c < -2$. As we will see, there is a simple space which is homeomorphic to the space of non-divergent orbits of Q_c, and it is much easier to understand the dynamics of this new space.

2 Itineraries and the sequence space Σ

Recall that for the family of functions Q_c, when $c < -2$, all the interesting dynamics occur on a single interval $I = [-p_+, p_+]$, where

$$p_+ = \frac{1 + \sqrt{1 - 4c}}{2}.$$

Moreover, the interval I has a subinterval A_1 in which all points leave I in a single iteration. The set Λ of points in I who's orbits never leave I therefore lies in the set $I - A_1$. This set A_1 partitions I into two closed sets, I_0 and I_1, and $\Lambda \in I_0 \cup I_1$. Now we can define what an itinerary of a point in Λ is.

Definition. Let $x \in \Lambda$. let Σ be the infinitely-dimensional space $\{0, 1\}^\omega$, that is, the space of points with an infinite number of coordinates, each of which only take on the values 0 or 1. The itinerary of x is a function from Λ to Σ

$$S(x) = (s_0, s_1, s_2, ...)$$

such that $\forall j \in \mathbb{N}$, $s_j = k$ if $Q_c^j(x) \in I_k$.

We will introduce a metric for Σ to better understand its topology. Let $\mathbf{s} = (s_0, s_1, s_2, ...)$ and $\mathbf{t} = (t_0, t_1, t_2, ...)$. The book defines a metric on Σ as

$$d(\mathbf{s}, \mathbf{t}) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i},$$

but we can easily come up with others. For example, a metric that looks more similar to the familiar Euclidean norm for finite-dimensional spaces is

$$d'(\mathbf{s},\mathbf{t}) = \sqrt{\sum_{i=0}^{\infty} \frac{(s_i - t_i)^2}{2^i}}.$$

The topology induced by both of these metrics on Σ is equivalent, but we will not prove this. The proof that d is in fact a metric is farely trivial and we will omit it. Do note that for each i the possible values $|s_i - t_i|$ can take are only 0 or 1, so that the infinite sum that defines d is bounded above by

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2,$$

so the metric d is well defined (i.e. does not diverge) $\forall \mathbf{s}, \mathbf{t} \in \Sigma$.

There is an important theorem, **the Proximity Theorem**, which states that for $\mathbf{s}$, $\mathbf{t}$ defined as before, $d(\mathbf{t},\mathbf{s}) < 1/2^n$ iff $s_i = t_i$ for $i \leq n$.

Proof. If $s_i = t_i$ for $i \leq n$, then

$$d(\mathbf{s},\mathbf{t}) = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}.$$

For the converse, if $s_j \neq t_j$ for some $j \leq n$, then we must have

$$d(\mathbf{s},\mathbf{t}) \geq \frac{1}{2^j} \geq \frac{1}{2^n},$$

so it must be the case that $s_i = t_i$ for $i \leq n$ if we require $d(\mathbf{t},\mathbf{s}) < 1/2^n$.

We can identify certain special subsets of Σ, like $M_0 = \{\mathbf{s} \in \Sigma | s_0 = 0\}$, and $M_1 = \{\mathbf{s} \in \Sigma | s_0 = 1\}$, which partition Σ. We can further partition M_0 into $M_{00} = \{\mathbf{s} \in \Sigma | s_0 = 0, s_1 = 0\}$ and $M_{01} = \{\mathbf{s} \in \Sigma | s_0 = 0, s_1 = 1\}$, and analogously define M_{10} and M_{11}, which partition M_1. Note that the Proximity Theorem tells us that the minimum distance between any point in M_0 and any other point in M_1 has to be 1, and the minimum distance between any point in $M_0 0$ and any point in $M_0 1$ is 1/2, so that these subsets are totally disconnected. In fact Σ is totally disconnected, which we can notice by inductively defining M_{000}, M_{001}, M_{100}, M_{101}, ... and showing that these subsets of Σ are always separated by a non-zero minimum distance by the Proximity Theorem. Recall that the Cantor Set is also totally disconnected, which hints the existence of a homeomorphism between the Cantor Set Λ and our itinerary space Σ.

3 The Shift Map σ

Definition. The shift map σ, from Σ to Σ is defined by

$$\sigma(s_0, s_1, s_2, ...) = (s_1, s_2, s_3, ...).$$

It is easy to find iterates of σ,

$$\sigma^n(s_0, s_1, s_2, ...) = (s_n, s_{n+1}, s_{n+2}, ...).$$

It is also easy to find the periodic points of σ. For example, the periodic points of period n have the form

$$(s_0, s_1, s_2, ..., s_{n-1}, s_0, s_1, ...).$$

Theorem. The shift map σ is a continuous map.

Proof. Suppose we are given $\mathbf{s} = (s_0, s_1, s_2, ...)$ and $\epsilon > 0$. Choose a value of n such that $1/2^n < \epsilon$, and let $\delta = 1/2^{n+1}$. Let $\mathbf{t} \in \Sigma$ be such that $d(\mathbf{s}, \mathbf{t}) < \delta$. Then, by the Proximity Theorem we must have that $s_i = t_i$ for $i < n + 1$. This implies that $\sigma(\mathbf{s})$ and $\sigma(\mathbf{t})$ agree in their first n coordinates, so a second application of the Proximity Theorem yields that $d(\sigma(\mathbf{s}), \sigma(\mathbf{t})) < \epsilon$, as desired.

Theorem. For every $x \in \Lambda$ it holds that $S(Q_c(x)) = \sigma(S(x))$.

Proof. It suffices to note that the itinerary of $Q_c(x)$ is the same as the itinerary of x except the first entry got deleted and the rest were shifted a coordinate back, but this is the same effect σ has on the itinerary of x.

4 The Homeomorphism Theorem

We have arrived to the main theorem of this lecture, which states that the topology of both Λ and Σ is essentially the same, so that analyzing the effects of σ on the points of Σ is the same as analyzing the effects of Q_c on the points of Λ, considering the last theorem. It is clearly easier to understand the former, so the goal of the lecture is complete.

Theorem. Λ and Σ are homeomorphic.

Proof. We need to show that the itinerary map S is one-to-one onto Σ, and that both the maps S and S^{-1} are continuous. We will only prove this for $c < -\frac{5+2\sqrt{5}}{4}$, although it holds for all $c < -2$.

One-to-one. Suppose $x, y \in \Lambda, x \neq y$, and $S(x) = S(y)$. This means x and y are always located in the same interval I_0 or I_1 for each iteration of Q_c. We know Q_c maps one-to-one into $I_0 \cup I_1$, and for the given values of c we also know that $|Q_c'(x)| > \mu > 1, \forall x \in I_0 \cup I_1$. For each n, Q_c^n takes the interval $[x, y]$ in a one-to-one fashion onto $[Q_c^n(x), Q_c^n(y)]$. Then the mean value theorem implies that

$$Q_c^n(y) - Q_c^n(x) \geq \mu^n(y - x),$$

but $\mu^n \to \infty$ as $n \to \infty$, implying that the distance between the n-th iterates of x and y diverges to infinity, even though they are always in the same interval I_0 or I_1. This contradiction establishes the proof that S maps into Σ one-to-one.

Onto. Let $Q_c^{-n}(J)$ denote the preimage of J under Q_c^n, and let $\mathbf{s} = (s_0, s_1, ...)$ be given. Define

$$I_{s_0,s_1,...,s_n} = \{x \in I | x \in I_{s_0}, Q_c(x) \in I_{s_1}, ..., Q_c^n(x) \in I_{s_n}\}.$$

This is the set of all the points in I who's itinerary coincides with that of x in the first n places. Rewriting $I_{s_0,s_1,...,s_n}$ in terms of preimages of other sets, and using identities of preimages, we can deduce that

$$I_{s_0,s_1,...,s_n} = I_{s_0} \cap Q_c^{-1}(I_{s_1}) \cap ... \cap Q_c^{-n}(I_{s_n}) = I_{s_0} \cap Q_c^{-1}((I_{s_1}) \cap ... \cap Q_c^{-(n-1)}(I_{s_n})),$$

$$I_{s_0,s_1,...,s_n} = I_{s_0} \cap Q_c^{-1}(I_{s_1,s_2,...,s_n}).$$

This implies that for every n the set $I_{s_0,s_1,...,s_n}$ is closed. We show this by induction. The set I_{s_0} is clearly closed, namely because it is I_0 or I_1. Suppose $I_{s_1,s_2,...,s_n}$ is closed. Then the preimage of $I_{s_1,s_2,...,s_n}$ is a pair of closed intervals, one in I_0 and the other in I_1. Hence,

$$I_{s_0,s_1,...,s_n} = I_{s_0} \cap Q_c^{-1}(I_{s_1,s_2,...,s_n})$$

is closed. These sets are also nested, for

$$I_{s_0,s_1,...,s_n} = I_{s_0,s_1,...,s_{n-1}} \cap Q_c^{-n}(I_{s_n}) \subset I_{s_0,s_1,...,s_{n-1}}$$

holds. This implies that

$$I_{s_0,s_1,...} = \bigcap_{n \geq 0} I_{s_0,s_1,...,s_n}$$

is non-empty (recall from your Analysis class that the intersection of arbitrary closed and nested sets is not empty). Hence, there is an $x \in I_{s_0,s_1,...}$ and it is such that $S(x) = \mathbf{s} = (s_0, s_1, ...)$. This implies every point in Σ is in the image of Λ under S, which proves the onto requirement.

Continuity. Let $x \in \Lambda$, $S(x) = (x_0, x_1, ...)$ and $\epsilon > 0$ be given. Choose a value of n such that $1/2^n < \epsilon$. Consider $I_{x_0,x_1,...,x_n}$, which is a closed interval as discussed earlier. Given the finite length of $I_{x_0,x_1,...,x_n}$, there clearly exists a δ such that if $|x - y| < \delta$ and $y \in \Lambda$ then $y \in I_{x_0,x_1,...,x_n}$. This implies that $S(y)$ and $S(x)$ have $x_i = y_i$ for $i \leq n$. The proximity theorem implies that $d(S(x), S(y)) \leq \epsilon$, which implies S is continuous. The proof that S^{-1} is continuous is very similar, so we omit it, and we are done proving the theorem.