Practice Quiz 1

18.100B R2 Fall 2010

Closed book, no calculators.

YOUR NAME: SOLUTIONS

This is a 30 minute in-class exam. No notes, books, or calculators are permitted. Point values are indicated for each problem. Do all the work on these pages.

GRADING	
1. ______	/15
2. ______	/20
3. ______	/15
4. ______	/20
TOTAL	/70

Problem 1. [5+5+5 points]

(a) Write down the definition of compactness in an arbitrary metric space.

$E \subset X$ is compact if, given any open cover $E \subset \bigcup_{\alpha \in A} \mathcal{U}_\alpha$ by open sets $\mathcal{U}_\alpha \subset X$ (with A any index set), one can find a finite subcover $E \subset \bigcup_{\alpha \in A'} \mathcal{U}_\alpha$, with $A' \subset A$ a finite subset.

(b) Prove that finite sets are always compact.

$E = \{e_1, \ldots, e_N\}$

Given an open cover $E \subset \bigcup_{\alpha \in A} \mathcal{U}_\alpha$,

for $i = 1..N$ pick $\alpha_i \in A$ s.t. $e_i \in \mathcal{U}_{\alpha_i}$,

then $A' := \{\alpha_1, \ldots, \alpha_N\} \subset A$ is finite

and $E \subset \bigcup_{\alpha \in A'} \mathcal{U}_\alpha$ is still a cover.

(c) Give an example of an infinite set that is not compact. (Show why it does not satisfy your definition in (a))

$E = \mathbb{N} \subset \mathbb{R}$

(because $\mathbb{N} \subset \bigcup_{n \in \mathbb{N}} B_{1/2}(n)$ is an open cover

with $m \in B_{1/2}(n) \Rightarrow m = n$, so if $\mathbb{N} \subset \bigcup_{n \in A'} B_{1/2}(n)$

then necessarily $m \in A'$ $\forall m$ $\Rightarrow A' = \mathbb{N}$ infinite)

Problem 2. [10+10 points]

(a) Let A and B be countable sets. Prove that $A \cup B$ is countable and that $A \cap B$ is at most countable, using the definition of countability.

By assumption we have $f: \mathbb{N} \to A$, $g: \mathbb{N} \to B$ bijections

Define a surjection $h: \mathbb{N} \to A \cup B$ by $h(2n-1) = f(n)$,
$h(2n) = g(n)$.

Then we can make it a bijection $h': \mathbb{N} \to A \cup B$
by $h'(1) = h(1)$, $h'(n+1) := h(m_n)$ with

$$m_n := \min \{ k \in \mathbb{N} \mid h(k) \notin \{h'(1), .., h'(n)\} \}$$

(m always exists because A infinite $\Rightarrow A \cup B$ infinite)

Similarly, define $f'(n) := f(m_n)$ with

$$m_n := \min \{ k \in \mathbb{N} \mid f(k) \in A \cap B \setminus \{f(1), .., f(n)\} \}.$$

If for some $n \in \mathbb{N}$, $A \cap B \setminus \{f(1), .., f(n)\} = \emptyset$, then
$A \cap B$ is finite ($\sim \{1, .., n\}$); otherwise
this defines a bijection $\mathbb{N} \to A \cap B$, so $A \cap B$ is
countable.

(b) Consider two subsets $S, T \subset \mathbb{R}$ and their sum

$$S + T := \{s + t \,|\, s \in S, t \in T\} \subset \mathbb{R}.$$

Show (from the definition of a supremum) that $\sup(S + T) = \sup S + \sup T$.

By definition,

$$\left.\begin{array}{ll} \bullet\ \forall s \in S & s \leq \sup S \\ \bullet\ \forall t \in T & t \leq \sup T \end{array}\right\} \Rightarrow \forall s+t \in S+T \quad s+t \leq \sup S + \sup T$$

So $\sup S + \sup T$ is an upper bound.

- $\forall \gamma < \sup S$ γ is not an upper bound, i.e. $\exists s \in S: \gamma < s$
- $\forall \beta < \sup T$ β is not an upper bound, i.e. $\exists t \in T: \beta < t$

$\Rightarrow$ Given $\alpha < \sup S + \sup T$ write $\alpha = \gamma + \beta$, $\begin{array}{l} \gamma < \sup S \\ \beta < \sup T \end{array}$

$$\left(\begin{array}{l} \gamma = \sup S - \frac{1}{2}(\sup S + \sup T - \alpha) \\ \beta = \sup T - \quad \text{———} \, " \, \text{———} \end{array} \right)$$

then $\exists s \in S, t \in T : \alpha = \gamma + \beta < s + t$,

$\Downarrow$

$s + t \in S + T$

so α is not an upper bound.

Problem 3. [5+5+5 points] Consider $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ as metric space with metric induced from the standard metric of $\mathbb{R}$.

a) What are the limit points of X?

only 0

(because $B_r(0) \cap E$ for $r > 0$ always contains some $\frac{1}{n} \neq 0$

all other points are isolated, $B_{1/2n}(\frac{1}{n}) \cap E = \{\frac{1}{n}\}$)

b) What are the closed subsets of X?

$\rightarrow$ finite subsets (which never have limit points)

$\rightarrow$ infinite subsets that contain 0

c) What are the compact subsets of X? Why?

finite subsets and infinite subsets that contain 0

because $X \subset \mathbb{R}$ is compact (bounded & closed) and the compact subsets of a compact set are exactly the closed subsets.

Problem 4. [20 points: $+4$ for each correct, -4 for each incorrect; no proofs required.]

a) For any open set $A \subset \mathbb{R}$, we have $int(\bar{A}) = A$.

TRUE FALSE

($int(\bar{A})$ does not contain isolated points of A)

b) Let V be the set of all functions $f : [0,1] \to \mathbb{R}$, and define $d(f,g) = |f(0) - g(0)|$. Then (V, d) is a metric space.

TRUE FALSE

(not definite : e.g. $f(x) = x$, $g(x) = x^2$ $\quad d(f,g) = |0-0| = 0$ but $f \neq g$)

c) If X is a compact metric space and $E \subset X$ is not compact, then E is not closed.

TRUE FALSE

(because closed subsets of compact sets are compact)

d) The set $\{(x,y) \in \mathbb{R}^2 | x + y \in \mathbb{Q}\}$ is countable.

TRUE FALSE

(there is an uncountable subset $\{(x,-x) \mid x \in \mathbb{R}\} \simeq \mathbb{R}$)

e) The set $\{(x,y) \in \mathbb{R}^2 | x + y \in \mathbb{Q}, x - y \in \mathbb{Q}\}$ is countable.

TRUE FALSE

($\left.\begin{matrix} x+y \in \mathbb{Q} \\ x-y \in \mathbb{Q} \end{matrix}\right\} \Rightarrow x, y \in \mathbb{Q}$, so the set is $\mathbb{Q} \times \mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \times \mathbb{Q}$, which is countable as countable union of countable sets)