Practice Quiz 2

18.100B R2 Fall 2010

Closed book, no calculators.

YOUR NAME: SOLUTIONS

This is a 30 minute in-class exam. No notes, books, or calculators are permitted. Point values are indicated for each problem. Do all the work on these pages.

Problem 1. [5+7+3 points]

Let (X, d) be a metric space and let $f\colon X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous maps.

(a) Suppose $f(x_0) > g(x_0)$. Show that there exists $r > 0$ such that $f(y) > g(y)$ for all $y \in B_r(x_0)$.

$R := f(x_0) - g(x_0) > 0$

f, g continuous, so with $\varepsilon = R/2$ find

- $\exists\ \delta_f > 0 : \quad f(B_{\delta_f}(x_0)) \subset (f(x_0) - R/2,\ f(x_0) + R/2)$
- $\exists\ \delta_g > 0 : \quad g(B_{\delta_g}(x_0)) \subset (g(x_0) - R/2,\ g(x_0) + R/2)$

Take $r = \min\{\delta_f, \delta_g\} > 0$, then

$$y \in B_r(x_0) \Rightarrow \begin{Bmatrix} f(y) > f(x_0) - R/2 \\ g(y) < g(x_0) + R/2 \end{Bmatrix} \Rightarrow f(y) - g(y) > R - R/2 - R/2 = 0$$

(b) Show that $s(x) := \max\{f(x), g(x)\}$ is a continuous map $s : X \to \mathbb{R}$.

To show. s continuous at any $x_0 \in X$

<u>3 cases</u>:

1) $f(x_0) > g(x_0)$: By (a), $s(x) = f(x)$ on $B_r(x_0)$ for some $r > 0$
f continuous $\Rightarrow$ s continuous at x_0

2) $f(x_0) < g(x_0)$: By (a), $s(x) = g(x)$ on $B_r(x_0)$ for some $r > 0$
g continuous $\Rightarrow$ s continuous at x_0

3) $f(x_0) = g(x_0)$: Given $\varepsilon > 0$, find $\delta_f, \delta_g > 0$ and $\delta = \min\{\delta_f, \delta_g\} > 0$

$$d(y, x_0) < \delta \Rightarrow \underbrace{f(B_\delta(x_0)) \cup g(B_\delta(x_0))}_{\cup \atop s(B_\delta(x_0))} \subset (s(x_0) - \varepsilon,\ s(x_0) + \varepsilon)$$

(c) Let $X = \mathbb{R}$ with the Euclidean metric. Is $s(x)$ as in (b) necessarily differentiable? (Give a proof or counterexample.)

<u>No</u>

$$\left.\begin{array}{l} f(x) = x \\ g(x) = 0 \end{array}\right\} \quad s(x) = \begin{cases} x & ; x \le 0 \\ 0 & ; x > 0 \end{cases}$$

s is not differentiable at $x_0 = 0$ because

$$\left.\begin{array}{l} \displaystyle\lim_{t\to 0^+} \frac{s(t)-s(0)}{t-0} = \lim_{t\to 0^+} \frac{0}{t} = \lim_{t\to 0^+} 0 = 0 \\ \displaystyle\lim_{t\to 0^-} \frac{s(t)-s(0)}{t-0} = \lim_{t\to 0} \frac{t}{t} = \lim_{t\to 0} 1 = 1 \end{array}\right\} \Rightarrow \lim_{t\to 0} \frac{s(t)-s(0)}{t-0} \text{ does not exist}$$

Problem 2. [10 points] Let $f : X \to Y$ be a continuous map between metric spaces. Show that for any connected subset $U \subset X$ the image $f(U) \subset Y$ is connected.

Suppose by contradiction $f(U) = A \cup B$ is separated:

$$\bar{A} \cap B = \emptyset, \quad A \cap \bar{B} = \emptyset, \quad A \neq \emptyset, \quad B \neq \emptyset$$

Then $U \subset f^{-1}(f(U))$ and hence

$$U = f^{-1}(A \cup B) \cap U = \left(f^{-1}(A) \cup f^{-1}(B)\right) \cap U = A' \cup B'$$

with $A' = f^{-1}(A) \cap U$, $B' = f^{-1}(B) \cap U$.

<u>Claim</u>: $U = A' \cup B'$ is separated in contradiction to assumption

<u>Proof</u>:
- $\exists a \in A \Rightarrow \exists a' \in U : f(a') = a \in A \Rightarrow \exists\, a' \in f^{-1}(A) \cap U = A'$
  $\Rightarrow A' \neq \emptyset$
- similarly $B' \neq \emptyset$
- if $x \in \overline{A'} \cap B'$ then $x \in U$, $f(x) \in B$, and
  $\exists (x_n)_{n \in \mathbb{N}} \subset U : f(x_n) \in A,\ x_n \to x$

  $$\left.\begin{array}{l} \text{by continuity of } f, \quad f(x_n) \to f(x) \in B \\ \text{by } f(x_n) \in A, \quad f(x) = \lim f(x_n) \in \bar{A} \end{array}\right\} \text{contradiction to } \bar{A} \cap B = \emptyset$$

  $\Rightarrow \overline{A'} \cap B' = \emptyset$
- similarly $A' \cap \overline{B'} = \emptyset$

(or as in Rudin 4.22)

Problem 3. [5+7+8 points]

(a) State the mean value theorem – in the exact version that you want to use in (b) and (c) below.

$f : [a,b] \to \mathbb{R}$ continuous, differentiable on (a,b)

$$\Rightarrow \exists x \in (a,b) : \quad \frac{f(b)-f(a)}{b-a} = f'(x)$$

(b) Suppose $g \colon \mathbb{R} \to \mathbb{R}$ is a differentiable function, and there is a constant $M > 0$ so that $|g'(x)| \leq M$ for all $x \in \mathbb{R}$. Show that for sufficiently small $\epsilon > 0$ the map $f(x) = x + \epsilon g(x)$ is an injective (one-to-one) map $f : \mathbb{R} \to \mathbb{R}$.

Pick $0 < \varepsilon < \frac{1}{M}$.

To check that f is injective, suppose by contradiction that $f(a) = f(b)$ for some $a < b$.

By mean value theorem, there exists $x \in (a,b)$ s.t.

$$0 = \frac{f(b)-f(a)}{b-a} = f'(x) = 1 + \varepsilon g'(x) \geq 1 - \varepsilon M > 0$$

$$\Rightarrow 0 > 0 \ ↯$$

(c) Let $f\colon (-1,1) \to \mathbb{R}$ be a continuous function, such that $f'(x)$ exists for all $x \neq 0$. Suppose that

$$\lim_{x\to 0} f'(x) = A$$

exists. Show that, in fact, f *is* differentiable at 0, with $f'(0) = A$.

By mean value theorem,

$\forall t>0 \ \exists x_t \in (0,t)$:
$\forall t<0 \ \exists x_t \in (t,0)$:
$$\frac{f(t)-f(0)}{t-0} = f'(x_t)$$

Consider $t_n \to 0$, then $\dfrac{f(t_n)-f(0)}{t_n-0} = f'(x_{t_n})$

where $0 < |x_{t_n}| < |t_n|$, hence $x_{t_n} \to 0$, and so by assumption ($\lim_{x\to 0} f'(x) = A$), $\dfrac{f(t_n)-f(0)}{t_n-0} = f'(x_{t_n}) \to A$.

This shows $\lim_{t\to 0} \dfrac{f(t)-f(0)}{t-0} = A$, as claimed.

Problem 4. [20 points: +4 for each correct, −4 for each incorrect; no proofs required.]

a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous with $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} = -\infty$, then for every $R \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $f(y) = R$.

TRUE FALSE

by intermediate value theorem
$\forall R \in \mathbb{R} \ \exists a<0, b>0 : f(a) > R, f(b) < R$
$\Rightarrow \exists a<y<b : f(y) = R$

b) A map $f : X \to Y$ is uniformly continuous if and only if

$$\forall N > 0 \ \exists R > 0 \ \forall y \in X : f(B_{R^{-1}}(y)) \subset B_{\frac{1}{N}}(f(y)).$$

TRUE FALSE

Given N, let $\varepsilon = 1/N$, which provides $\delta > 0$, then take $R > \delta^{-1}$

Given ε, pick $N > \varepsilon^{-1}$, which provides $R > 0$, then take $\delta \in (0, R^{-1})$

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall y \ f(B_\delta(y)) \subset B_\varepsilon(f(y))$

c) Let $f : R \to \mathbb{R}$ be continuous, then $g(x) = xf(x)$ is differentiable at $x = 0$.

TRUE FALSE

$\lim_{t\to 0} \frac{g(t)-g(0)}{t-0} = \lim_{t\to 0} \frac{t\,g(t)}{t} = \lim_{t\to 0} g(t) = g(0)$ exists

d) Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $|f'(x)| \le C$ for all $x \in \mathbb{R}$ and some $C > 0$. Then f is uniformly continuous.

TRUE FALSE

$d(f(x), f(y)) = |f(x) - f(y)|$
$= |f'(z)(x-y)|$ for some $z \in \mathbb{R}$
$\le C\, d(x,y)$
So for $\varepsilon > 0$ take $\delta = \varepsilon / C$.

e) Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and uniformly continuous. Then $|f'|$ is bounded.

TRUE FALSE

<u>Idea</u>: f can vary little with large f' by fast oscillation

$f(x) = \frac{1}{x^2+1} \cdot \sin x^4$ is diff'ble and uniformly continuous: Given $\varepsilon > 0$
- find $R > 0$ s.t. $\forall |x| > R \ |f(x)| < \varepsilon/2$
- find $\delta > 0$ from continuity on $[-R, R]$

<u>but</u> $f'(x) = \frac{3x^3 \sin x^4 (x^2+1) - \sin x^4 \, 2x}{(x^2+1)^2}$

so for $x_n = \sqrt[4]{n\pi + \pi/2} \to \infty$ get $f'(x_n) \to \infty$