Practice Quiz 4

18.100B R2 Fall 2010

Closed book, no calculators.

YOUR NAME: SOLUTIONS

This is a 30 minute in-class exam. No notes, books, or calculators are permitted. Point values are indicated for each problem. Do all the work on these pages.

Problem 1. [15 points] Fix $n \in \mathbb{N}$ and let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{2^n} & \text{if } x = \frac{k}{2^n} \text{ for } k \text{ odd }, 0 < k < 2^n, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is Riemann integrable on $[0,1]$, and that $\int_0^1 f(x)\,dx = 0$.

- $$L(f, P=(x_i)) = \sum_{i=1}^{n} \underbrace{\inf_{x \in [x_{i-1}, x_i]} f(x)}_{\geq 0} \underbrace{(x_i - x_{i-1})}_{>0} \geq 0$$

$$\Rightarrow L(f) = \sup_P L(f,P) \geq 0$$

- $$U(f, P=(x_i)) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x) \, (x_i - x_{i-1})$$

Given $\varepsilon > 0$, pick a partition

$$P^\varepsilon = \left(0, \frac{1}{2^n} - \delta, \frac{1}{2^n} + \delta, \frac{3}{2^n} - \delta, \frac{3}{2^n} + \delta, \dots, \frac{2^n - 1}{2^n} - \delta, \frac{2^n-1}{2^n} + \delta, 1\right)$$

with $\delta < \frac{1}{2^n}$ $\left(\text{hence } \frac{1}{2^n} + \delta < \frac{3}{2^n} - \delta\right)$ and $\delta \leq \varepsilon$, then

$$U(f, P^\varepsilon) = \underbrace{\sum_{k=1}^{2^{n-1}} \underset{\substack{\| \\ 1/2^n}}{\sup f} \cdot \underset{\substack{\| \\ 2\delta}}{\Delta x_i}}_{\text{from intervals } \left[\frac{k}{2^n} - \delta, \frac{k}{2^n} + \delta\right]} + \underbrace{0}_{\text{from other intervals}} = \frac{2^n - 1}{2^{n-1}} \delta < 2\delta < \varepsilon$$

Together with Rudin ($L(f) \leq U(f)$) this shows

$$0 \leq L(f) \leq U(f) \leq \varepsilon \quad \forall \varepsilon > 0 \Rightarrow L(f) = U(f) = 0$$

$$\Rightarrow f \text{ integrable}, \int_0^1 f\,dx = L(f) = 0$$

Problem 2. [10 points] Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, nonnegative (i.e. $f(x) \geq 0$ for all $x \in [a, b]$), and $\int_a^b f(x)dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

f continuous $\Rightarrow$ integrable $\Rightarrow$ $\underset{\displaystyle \sup_P L(f,P)}{\underset{\shortparallel}{L(f)}} = 0$

$\Rightarrow \forall$ partitions P $L(f,P) \leq 0$

Suppose by contradiction $f(x_0) > 0$ for some $x_0 \in [a,b]$, then by continuity find $\delta > 0$ s.t.

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \tfrac{1}{2} f(x_0)$$

$$\Rightarrow f(x) \geq f(x_0) - \tfrac{1}{2} f(x_0) = \tfrac{1}{2} f(x_0)$$

Now consider any equidistant partition P with $\Delta x < \delta$, then

$$L(f,P) = \sum_{i=1}^{n} \underbrace{\inf_{[x_{i-1}, x_i] \ni x} f(x)}_{\geq 0 \text{ since } f(x) \geq 0} \Delta x \geq \inf_{x \in [x_{i_0 - 1}, x_{i_0}] \ni x_0} f(x) \cdot \Delta x$$

$$\geq \tfrac{1}{2} f(x_0) \cdot \Delta x > 0$$

just looking at an interval that contains x_0

in contradiction to $L(f,P) \leq 0$

Problem 3. [5+7+8 points]

(a) Consider the sequence of partial sums

$$f_n(x) = \sum_{k=1}^{n} e^{-kx}\cos(kx).$$

For any $a > 0$ show that f_n converges uniformly on $[a, \infty)$.

$m > n$

$$\|f_n - f_m\|_\infty = \left\|\sum_{k=n+1}^{m} e^{-kx}\cos(kx)\right\|_\infty \leq \sum_{k=n+1}^{m} \sup_{x \geq a} e^{-kx}|\cos kx|$$

$$\leq \sum_{k=n+1}^{m} e^{-ka} \xrightarrow[n\to\infty]{} 0$$

since $\sum e^{-ka} = \sum (e^{-a})^k$ converges due to $|e^{-a}| < 1$

This shows uniform convergence by the "Cauchy criterion" (in Rudin).

(b) Let $f(x)$ denote the limit of the sequence $f_n(x)$ in (a). Show that $f(x)$ is continuous on $(0, \infty)$.

To show that f is continuous at $x_0 \in (0,\infty)$ note that

- each f_n is continuous on $[x_0/2, 2x_0]$
- $f_n \to f$ uniformly on $[x_0/2, 2x_0]$ by (a)

So, by Rudin, f is continuous on $[x_0/2, 2x_0]$, which contains x_0.

(c) Using the function $f : (0,\infty) \to \mathbb{R}$ from (b), find (and prove) an explicit upper bound for $|\int_1^\infty f(x)dx|$, such as 2 or $\frac{e}{e-1}$. (Hint: You only need to integrate e^{-nx} for this estimate.)

$$\int_1^\infty f(x)\,dx \underset{\text{by def}^{\underline{n}}}{=} \lim_{b\to\infty} \int_1^b f(x)\,dx \underset{\text{by Rudin}}{=} \lim_{b\to\infty}\lim_{n\to\infty} \int_1^b f_n(x)\,dx$$

$$\lim_{n\to\infty} \int_1^b f_n(x)\,dx \underset{\substack{\text{linearity of}\\ \text{integral}}}{=} \lim_{n\to\infty} \sum_{k=1} \int_1^b \underbrace{e^{-kx}\cos kx}_{1\cdot 1 \le e^{-kx}}\,dx$$

exists by comparison with the absolutely convergent series $\sum_{k=1}^\infty e^{-k}$,

and

$$\left|\int_1^b f(x)dx\right| = \left|\lim_{n\to\infty} \int_1^b f_n(x)\,dx\right| \le \sum_{k=1}^n \int_1^b e^{-kx}\,dx = \sum_{k=1}^n \left[\frac{-1}{k}e^{-kx}\right]_1^b \le \sum_{k=1}^n \frac{1}{k}e^{-k}$$

$$\le \sum_{k=1}^n \left(\frac{1}{e}\right)^k \le \sum_{k=1}^\infty (1/e)^k = \frac{1}{1-1/e} = \frac{e}{e-1}$$

Similarly, $\lim_{b\to\infty} \int_1^b f\,dx$ exists since for $b' \geq b$

$$\left| \int_1^{b'} f\,dx - \int_1^b f\,dx \right| = \lim_{n\to\infty} \left| \int_b^{b'} f_n\,dx \right| \le \sum_{k=1}^\infty \frac{1}{k}(e^{-bk} - e^{-b'k}) \le \sum_{k=1}^\infty (e^{-b})^k = \frac{1}{1-e^{-b}}$$

converges to 0 as $b\to\infty$. (Hence the same holds for any sequence $b_i \to \infty$, making $\int_1^{b_i} f\,dx$ a Cauchy sequence. Completeness of $\mathbb{R}$ then implies convergence as $i\to\infty$; and the limit for all sequences $b_i \to\infty$ is the same since otherwise one could make a divergent (oscillating) sequence.)

So $\int_1^\infty f\,dx$ exists, and $\int_1^\infty f\,dx = \lim_{b\to\infty} \int_1^b f\,dx \le \lim_{b\to\infty} \frac{e}{e-1} = \frac{e}{e-1}$.

Problem 4. [20 points: $+4$ for each correct true/false, -4 for each incorrect true/false; you can opt for 'unsure' and gain up to $+2$ for giving your thoughts.]

a) Suppose $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b). Then f is Riemann integrable.

TRUE UNSURE FALSE

[differentiable implies continuous, but not bounded - e.g. $f(x) = x^{-1}$ on $[0,1]$ is diff'ble on $(0,1)$]

b) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. Then the function $F : [a,b] \to \mathbb{R}$ given by $F(x) = \int_a^x f(t)dt$ is continuous.

TRUE UNSURE FALSE

[see Rudin]

c) If $f : [a,b] \to \mathbb{R}$ is Riemann integrable and satisfies $f(x) \leqslant 0$ for all $x \in [a,b] \cap (\mathbb{R} \setminus \mathbb{Q})$, then $\int_a^b f(x)dx \leqslant 0$.

TRUE UNSURE FALSE

[$L(f,P) = \sum \underbrace{\inf_{[x_{i-1},x_i] \ni x} f(x)}_{\leq 0} \cdot \Delta x_i \leq 0 \Rightarrow L(f) = \int f dx \leq 0$
≤ 0 since any interval contains $x \in \mathbb{R} \setminus \mathbb{Q}$]

d) If $f_n : [a,b] \to \mathbb{R}$ is a sequence of continuous functions, and $f_n \to f$ converges uniformly, then the limit f is uniformly continuous.

TRUE UNSURE FALSE

[the limit is continuous by Rudin 7...., and uniformly continuous since $[a,b]$ is compact]

e) If $f_n : [a,b] \to \mathbb{R}$ is a sequence of almost everywhere continuous functions, and $f_n \to f$ converges uniformly, then the limit f is almost everywhere continuous.

TRUE UNSURE FALSE

[almost everywhere continuous $\Longleftrightarrow$ Riemann integrable
so result follows from "$f_n \in \mathcal{R}, \|f_n - f\|_\infty \to 0 \Rightarrow f \in \mathcal{R}$"]

18.100B Analysis I
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