Problem Set 7 Solutions, 18.100C, Fall 2012

October 31, 2012

1

We wish to define $\varpi/2$ as the smallest zero of $\cos(x)$, i.e. as the positive real number such that $\cos(\varpi/2) = 0$ and for $0 < x < \varpi/2$, $\cos(x) \neq 0$. Clearly such a number, if it exists, is unique; if b and b' both satisfy the condition, then $b \leq b'$ and $b' \leq b$, so $b = b'$. So we just have to prove existence.

Assume first that there exists some $a > 0$ such that $\cos(a) = 0$. Consider the set $E = \cos^{-1}(0) \cap [0, a]$. Since cos is continuous E is a closed subset of the compact set $[0, a]$. E is also non-empty, since $a \in E$. Let $b = \inf E$; since E is closed $b \in E$, i.e. $\cos(b) = 0$. If $\cos(x) = 0$, then either $x > a > b$, or $x \leq a$ and $x \in E$, so $x \geq b$. So we have $\varpi/2 = b$. Note that the only fact about cos that we used is that it is continuous; indeed, every continuous function on $[0, \infty)$ with at least one zero has a smallest zero, although unlike the case of cos that zero will not always be isolated.

It remains to show that cos has some positive zero. Note that $\cos(0) = 1$. We will show in the next paragraph that $\cos(2) < 0$. Since cos is continuous, we can apply the Intermediate Value Theorem to the interval $[0, 2]$ to conclude that there exists $a \in (0, 2)$ with $\cos(a) = 0$.

Let $a_k = (-1)^k 2^{2k}/(2k)!$. Then $\cos(2) = \sum_{k=0}^{\infty} a_k$. Note that

$$|a_{k+1}| = \frac{4}{(2k+1)(2k+2)} |a_k|$$

So as long as $k > 0$, $|a_k| > |a_{k+1}|$. Hence, after the first two terms, $|a_k|$ is an alternating sequence of stricly decreasing abosulte value, with the absolute value converging to zero. However, as we showed on a previous homework, all the odd partial sums of such a series are bigger than the sum of the whole series. In particular, we have

$$\cos(2) = \sum_{k=0}^{\infty} a_k < \sum_{k=0}^{2} a_k = 1 - \frac{2^2}{2} + \frac{2^4}{4!} = -\frac{1}{3} < 0$$

2

Let $f : I \to I$ be a continuous function. Define $g : I \to \mathbb{R}$ by $g(x) = x - f(x)$. Then

$$g(0) = 0 - f(0) \leq 0 \quad \text{and} \quad 0 \leq 1 - f(1) = g(1).$$

If equality holds in either case then we are done. Otherwise $g(0) < 0 < g(1)$. One can check that g is continuous (use Theorem 4.6 and 4.4a)) and so we can apply Theorem 4.23 to find a point $x \in (0, 1)$ such that $g(x) = 0$, i.e. $f(x) = x$.

3

Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that

$$|f(x) - f(y)| \leq (x - y)^2 \quad \text{for all} \;\; x, y \in \mathbb{R}.$$

Fix $x \in \mathbb{R}$. Then for $y \neq x$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y| \to 0 \quad \text{as} \quad y \to x.$$

Thus $f'(x) = 0$. By Theorem 5.11b), f is constant.

18.100C Real Analysis
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