Lecture 12

So far, we have been studying only the Riemann integral. However, there is also the Lebesgue integral. These are the two basic integral theories. The Riemann integral is very intuitive and is usually adequate for problems that usually come up. The Lebesgue integral is not as intuitive, but it can handle more general problems. We do not encounter these problems in geometry or physics, but we would in probability and statistics. You can learn more about Lebesgue integrals by taking Fourier Analysis (18.103) or Measure and Integration (18.125). We do not study the Lebesgue integral.

Let S be a bounded subset of $\mathbb{R}^n$.

Theorem 3.17. *If the boundary of S is of measure zero, then the constant function 1 is R. integrable over S. The converse is also true.*

Proof. Let Q be a rectangle such that $\text{Int } Q \supset \bar{S}$. Define

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases} \tag{3.110}$$

The constant function 1 is integrable over S if and only if the function 1_S is integrable over Q. The function 1_S is integrable over Q if the set of points D in Q where 1_S is discontinuous is of measure zero. If so, then

$$\int_Q 1_S = \int_S 1. \tag{3.111}$$

Let $x \in Q$.

1. If $x \in \text{Int } S$, then $1_S = 1$ in a neighborhood of x, so 1_S is continuous at x.

2. If $x \in \text{Ext } S$, then $1_S = 0$ in a neighborhood of x, so 1_S is continuous at x.

3. If $x \in \text{Bd } X$, then in every neighborhood U of x there exists points in $\text{Ext } S$ where $1_S = 0$ and points in $\text{Int } S$ where $1_S = 1$. So, 1_S is discontinuous at x.

Thus, D is the boundary of S, $D = \text{Bd } S$. Therefore, the function 1_S is integrable if and only if $\text{Bd } S$ is of measure zero. $\square$

3.7 Improper Integrals

Definition 3.18. The set S is *rectifiable* if the boundary of S is of measure zero. If S is rectifiable, then

$$v(S) = \int_S 1. \tag{3.112}$$

Let us look at the properties of $v(S)$:

1. Monotonicity: If S_1 and S_2 are rectifiable and $S_1 \subseteq S_2$, then $v(S_1) \leq v(S_2)$.
2. Linearity: If S_1, S_2 are rectifiable, then $S_1 \cup S_2$ and $S_1 \cap S_2$ are rectifiable, and
$$v(S_1 \cup S_2) = v(S_1) + v(S_2) - v(S_1 \cap S_2). \tag{3.113}$$
3. If S is rectifiable, then $v(S) = 0$ if and only if S is of measure zero.
4. Let $A = \text{Int } S$. If S is rectifiable, then A is rectifiable, and $v(S) = v(A)$.

The first two properties above are special cases of the theorems that we proved last lecture:

1.
$$\int_{S_1} 1 \leq \int_{S_2} \text{ if } S_1 \subseteq S_2. \tag{3.114}$$
2.
$$\int_{S_1 \cup S_2} 1 = \int_{S_1} 1 + \int_{S_2} 1 - \int_{S_1 \cap S_2} 1. \tag{3.115}$$

To see the the third and fourth properties are true, we use some previous results. Let Q be a rectangle, and let $f : Q \to \mathbb{R}$ be R. integrable. We proved the following two theorems:

Theorem A. *If $f \geq 0$ and $\int_Q f = 0$, then $f = 0$ except on a set of measure zero.*

Theorem B. *If $f = 0$ except on a set of measure zero, then $\int_S f = 0$.*

Property 3. above is a consequence of Theorem A with $f = 1_S$.
Property 4. above is a consequence of Theorem B with $f = 1_S - 1_A$.
We are still lacking some simple criteria for a bounded set to be integrable. Let us now work on that.

Let S be a bounded set, and let $f : S \to \mathbb{R}$ be a bounded function. We want simple criteria on S and f such that f to be integrable over S.

Theorem 3.19. *If S is rectifiable and $f : S \to \mathbb{R}$ is bounded and continuous, then f is R. integrable over S.*

Proof. Let Q be a rectangle such that $\text{Int } Q \supset \bar{S}$. Define $f_S : Q \to \mathbb{R}$ by
$$f_S(x) = \begin{cases} f(x) & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases} \tag{3.116}$$

By definition, f is integrable over S if and only if f_S is integrable over Q. If so then $\int_S f = \int_Q f_S$.

Let D be the set of points in Q where f_S is discontinuous. Then f_S is integrable over Q if and only if D is of measure zero. What is D?

1. If $x \in \text{Int } S$, then $f_S = f$ in a neighborhood of x, so f_S is continuous at x.

2. If $x \in \text{Ext } S$, then $f_S = 0$ in a neighborhood of x, so f_S is continuous at x.

So, we know that $D \subseteq \text{Bd } S$. Because S is rectifiable, the boundary of S has measure zero, so D has measure zero. Thus, f_S is R. integrable, and therefore so is f. □

Theorem 3.20. *Let A be an open subset of $\mathbb{R}^n$. There exists a sequence of compact rectifiable sets $C_N, N = 1, 2, 3, \ldots$ such that*

$$C_N \subseteq \text{Int } C_{N+1} \tag{3.117}$$

and

$$\bigcup C_N = A. \tag{3.118}$$

Definition 3.21. The set $\{C_N\}$ is called an *exhaustion* of A.

Proof. Take the complement of A, namely $B = \mathbb{R}^n - A$. Define $d(x, B) = \inf_{y \in B}\{|x - y|\}$. The function $d(x, B)$ is a continuous function of x (the theorem for this is in section 4 of Munkres). Let

$$D_N = \{x \in A : d(x, B) \geq 1/N \text{ and } |x| \leq N\}. \tag{3.119}$$

The set D_N is compact. It is easy to check that $D_N \subseteq \text{Int } D_{N+1}$.

Claim.

$$\bigcup D_N = A. \tag{3.120}$$

Proof. Let $x \in A$. The set A is open, so there exists $\epsilon > 0$ such that the set $\{y \in \mathbb{R}^n : |y - x| \leq \epsilon\}$ is contained in A. So, $d(x, B) \geq \epsilon$.

Now, choose N such that $1/N < \epsilon$ and such that $|x| < N$. Then, by definition, $x \in D_N$. Therefore $\cup D_N = A$. □

So, the D_N's satisfy the right properties, except they are not necessarily rectifiable. We can make them rectifiable as follows.

For every $p \in D_N$, let Q_p be a rectangle with $p \in \text{Int } Q_p$ and $Q_p \subseteq \text{Int } D_{N+1}$. Then the collection of sets $\{\text{Int } Q_p : p \in D_N\}$ is an open cover of D_N. By the H-B Theorem, there exists a finite subcover $\text{Int } Q_{p_1}, \ldots, \text{Int } Q_{p_r}$. Now, let

$$C_N = Q_{p_1} \cup \cdots \cup Q_{p_r}. \tag{3.121}$$

Then $C_N \subseteq \text{Int } D_N \subseteq \text{Int } C_{N+1}$. □