# Lecture 18

We begin with a quick review of permutations (from last lecture).

A permutation of order $k$ is a bijective map $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$. We denote by $S_k$ the set of permutations of order $k$.

The set $S_k$ has some nice properties. If $\sigma \in S_k$, then $\sigma^{-1} \in S_k$. The inverse permutation $\sigma^{-1}$ is defined by $\sigma^{-1}(j) = i$ if $\sigma(i) = j$. Another nice property is that if $\sigma, \tau \in S_k$, then $\sigma\tau \in S_k$, where $\sigma\tau(i) = \sigma(\tau(i))$. That is, if $\tau(i) = j$ and $\sigma(j) = k$, then $\sigma\tau(i) = k$.

Take $1 \leq i < j \leq k$, and define

$$\tau_{i,j}(i) = j \tag{4.40}$$

$$\tau_{i,j}(j) = i \tag{4.41}$$

$$\tau_{i,j}(\ell) = \ell, \ell \neq i, j. \tag{4.42}$$

The permutation $\tau_{i,j}$ is a transposition. It is an elementary transposition of $j = i+1$. Last time we stated the following theorem.

**Theorem 4.13.** *Every permutation $\sigma$ can be written as a product*

$$\sigma = \tau_1 \tau_2 \cdots \tau_r, \tag{4.43}$$

*where the $\tau_i$'s are elementary transpositions.*

In the above, we removed the symbol $\circ$ denoting composition of permutations, but the composition is still implied.

Last time we also defined the sign of a permutation

**Definition 4.14.** The sign of a permutation $\sigma$ is $(-1)^\sigma = (-1)^r$, where $r$ is as in the above theorem.

**Theorem 4.15.** *The above definition of sign is well-defined, and*

$$(-1)^{\sigma\tau} = (-1)^\sigma (-1)^\tau. \tag{4.44}$$

All of the above is discussed in the Multi-linear Algebra Notes.

Part of today's homework is to show the following two statements:

1. $|S_k| = k!$. The proof is by induction.

2. $(-1)^{\tau_{i,j}} = -1$. Hint: use induction and $\tau_{i,j} = (\tau_{j-1,j})(\tau_{i,j-1})(\tau_{j-1,j})$, with $i < j$.

We now move back to the study of tensors. Let $V$ be an $n$-dimensional vector space. We define

$$V^k = \underbrace{V \times \cdots \times V}_{k \text{ factors}}. \tag{4.45}$$

We define $\mathcal{L}^k(v)$ to be the space of all $k$-linear functions $T : V^k \to \mathbb{R}$. If $T_i \in \mathcal{L}^{k_i}, i = 1, 2$, and $k = k_1 + k_2$, then $T_1 \otimes T_2 \in \mathcal{L}^k$. Decomposable $k$-tensors are of the form $T = \ell_1 \otimes \cdots \otimes \ell_k$, where each $\ell_i \in \mathcal{L}^1 = V^*$. Note that $\ell_1 \otimes \cdots \otimes \ell_k(v_1, \dots, v_k) = \ell_1(v_1) \dots \ell_k(v_k)$.

We define a permutation operation on tensors. Take $\sigma \in S_k$ and $T \in \mathcal{L}^k(V)$.

**Definition 4.16.** We define the map $T^\sigma : V^k \to \mathbb{R}$ by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}). \tag{4.46}$$

Clearly, $T^\sigma \in \mathcal{L}^k(V)$. We have the following useful formula:

**Claim.**

$$(T^\sigma)^\tau = T^{\tau\sigma}. \tag{4.47}$$

*Proof.*

$$\begin{aligned} T^{\tau\sigma}(v_1, \dots, v_k) &= T(v_{\sigma^{-1}(\tau^{-1}(1))}, \dots, v_{\sigma^{-1}(\tau^{-1}(k))}) \\ &= T^\sigma(v_{\tau^{-1}(1)}, \dots, v_{\tau^{-1}(k)}) \\ &= (T^\sigma)^\tau(v_1, \dots, v_k). \end{aligned} \tag{4.48}$$

□

Let us look at what the permutation operation does to a decomposable tensor $T = \ell_1 \otimes \cdots \otimes \ell_k$.

$$T^\sigma(v_1, \dots, v_k) = \ell_1(v_{\sigma^{-1}(1)}) \dots \ell_k(v_{\sigma^{-1}(k)}). \tag{4.49}$$

The $i$th factor has the subscript $\sigma^{-1}(i) = j$, where $\sigma(j) = i$, so the the $i$th factor is $\ell_{\sigma(j)}(v_j)$. So

$$\begin{aligned} T^\sigma(v_1, \dots, v_k) &= \ell_{\sigma(1)}(v_1) \dots \ell_{\sigma(k)}(v_k) \\ &= (\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)})(v_1, \dots, v_k). \end{aligned} \tag{4.50}$$

To summarize,

$$\begin{cases} T = \ell_1 \otimes \cdots \otimes \ell_k \\ T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}. \end{cases} \tag{4.51}$$

**Proposition 4.17.** *The mapping $T \in \mathcal{L}^k \to T^\sigma \in \mathcal{L}^k$ is linear.*

We leave the proof of this as an exercise.

**Definition 4.18.** A tensor $T \in \mathcal{L}^k(V)$ is *alternating* if $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.

**Definition 4.19.** We define

$$\mathcal{A}^k(V) = \text{ the set of all alternating } k\text{-tensors}. \tag{4.52}$$

By our previous claim, $\mathcal{A}^k$ is a vector space.

The *alternating operator* Alt can be used to create alternating tensors.

**Definition 4.20.** Given a $k$-tensor $T \in \mathcal{L}^k(V)$, we define the *alternating operator* $\mathrm{Alt} : \mathcal{L}^k(V) \to \mathcal{A}^k(V)$ by

$$\mathrm{Alt}\,(T) = \sum_{\tau \in S_k} (-10)^\tau T^\tau. \tag{4.53}$$

**Claim.** *The alternating operator has the following properties:*

1. $\mathrm{Alt}\,(T) \in \mathcal{A}^k(V)$,
2. *If* $T \in \mathcal{A}^k(V)$, *then* $\mathrm{Alt}\,(T) = k!T$,
3. $\mathrm{Alt}\,(T^\sigma) = (-1)^\sigma \mathrm{Alt}\,(T)$,
4. *The map* $\mathrm{Alt} : \mathcal{L}^k(V) \to \mathcal{A}^k(V)$ *is linear.*

*Proof.* 1.

$$\mathrm{Alt}\,(T) = \sum_\tau (-1)^\tau T^\tau, \tag{4.54}$$

so

$$\begin{aligned}\mathrm{Alt}\,(T)^\sigma &= \sum_\tau (-1)^\tau (T^\tau)^\sigma \\ &= \sum_\tau (-1)^\tau T^{\sigma\tau} \\ &= (-1)^\sigma \sum_{\sigma\tau} (-1)^{\sigma\tau} T^{\sigma\tau} \\ &= (-1)^\sigma \mathrm{Alt}\,(T).\end{aligned} \tag{4.55}$$

2.

$$\mathrm{Alt}\,(T) = \sum_\tau (-1)^\tau T^\tau, \tag{4.56}$$

but $T^\tau = (-1)^\tau T$, since $T \in \mathcal{A}^k(V)$. So

$$\begin{aligned}\mathrm{Alt}\,(T) &= \sum_\tau (-1)^\tau (-1)^\tau T \\ &= k!T.\end{aligned} \tag{4.57}$$

3.

$$\begin{aligned}\mathrm{Alt}\,(T^\sigma) &= \sum_\tau (-1)^\tau (T^\sigma)^\tau \\ &= \sum_\tau (-1)^\tau T^{\tau\sigma} \\ &= (-1)^\sigma \sum_{\tau\sigma} (-1)^{\tau\sigma} T^{\tau\sigma} \\ &= (-1)^\sigma \mathrm{Alt}\,(T).\end{aligned} \tag{4.58}$$

4. We leave the proof as an exercise.

$\square$

Now we ask ourselves: what is the dimension of $\mathcal{A}^k(V)$? To answer this, it is best to write a basis.

Earlier we found a basis for $\mathcal{L}^k$. We defined $e_1, \dots, e_n$ to be a basis of $V$ and $e_1^*, \dots, e_n^*$ to be a basis of $V^*$. We then considered multi-indices $I = (i_1, \dots, i_k), 1 \leq i_r \leq n$ and defined $\{e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*,\ I \text{ a multi-index}\}$ to be a basis of $\mathcal{L}^k$. For any multi-index $J = (j_1, \dots, j_k)$, we had

$$e_I^*(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \tag{4.59}$$

**Definition 4.21.** A multi-index $I = (i_1, \dots, i_k)$ is *repeating* if $i_r = i_s$ for some $r < s$.

**Definition 4.22.** The multi-index $I$ is *strictly increasing* if $1 \leq i_1 < \ldots < i_k \leq n$.

**Notation.** Given $\sigma \in S_k$ and $I = (i_1, \dots, i_k)$, we denote $I^\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)})$.

**Remark.** If $J$ is a non-repeating multi-index, then there exists a permutation $\sigma$ such that $J = I^\sigma$, where $I$ is strictly increasing.

$$e_J^* = e_{I^\sigma}^* = e_{\sigma(i_1)}^* \otimes \cdots \otimes e_{\sigma(i_k)}^* = (e_I^*)^\sigma. \tag{4.60}$$

Define $\psi_I = \text{Alt}\,(e_I^*)$.

**Theorem 4.23.** *1. $\psi_{I^\sigma} = (-1)^\sigma \psi_I$,*

*2. If $I$ is repeating, then $\psi_I = 0$,*

*3. If $I, J$ are strictly increasing, then*

$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & \textit{if } I = J, \\ 0 & \textit{if } I \neq J. \end{cases} \tag{4.61}$$

*Proof.* 1.

$$\begin{aligned} \psi_{I^\sigma} &= \text{Alt } e_{I^\sigma}^* \\ &= \text{Alt}\,((e_I^*)^\sigma) \\ &= (-1)^\sigma \text{ Alt } e_I^* \\ &= (-1)^\sigma \psi_I. \end{aligned} \tag{4.62}$$

2. Suppose that $I$ is repeating. Then $I = I^\tau$ for some transposition $\tau$. So $\psi_I = (-1)^\tau \psi_I$. But (as you proved in the homework) $(-1)^\tau = -1$, so $\psi_I = 0$.

3.

$$\begin{aligned}\psi_I &= \mathrm{Alt}\,(e_I^*)\\ &= \sum_\tau (-1)^\tau e_{I^\tau}^*,\end{aligned} \tag{4.63}$$

so

$$\psi_I(e_{j_1},\dots,e_{j_k}) = \sum_\tau (-1)^\tau \underbrace{e_{I^\tau}^*(e_{j_1},\dots,e_{j_k})}_{\begin{cases}1 & \text{if } I^\tau = J,\\ 0 & \text{if } I^\tau \neq J.\end{cases}} \tag{4.64}$$

But $I^\tau = J$ only if $\tau$ is the identity permutation (because both $I^\tau$ and $J$ are strictly increasing). The only non-zero term in the sum is when $\tau$ is the identity permutation, so

$$\psi_I(e_{j_1},\dots,e_{j_k}) = \begin{cases}1 & \text{if } I = J,\\ 0 & \text{if } I \neq J.\end{cases} \tag{4.65}$$

□

**Corollary 5.** *The alternating $k$-tensors $\psi_I$, where $I$ is strictly increasing, are a basis of $\mathcal{A}^k(V)$.*

*Proof.* Take $T \in \mathcal{A}^k(V)$. The tensor $T$ can be expanded as $T = \sum c_I e_I^*$. So

$$\begin{aligned}\mathrm{Alt}\,(T) &= k! \sum c_I \,\mathrm{Alt}\,(e_I^*)\\ &= k! \sum c_I \psi_I.\end{aligned} \tag{4.66}$$

If $I$ is repeating, then $\psi_I = 0$. If $I$ is non-repeating, then $I = J^\sigma$, where $J$ is strictly increasing. Then $\psi_I = (-1)^\sigma \psi_J$.

So, we can replace all multi-indices in the sum by strictly increasing multi-indices,

$$T = \sum a_I \psi_I, \ \ I\text{'s strictly increasing.} \tag{4.67}$$

Therefore, the $\psi_I$'s span $\mathcal{A}^k(V)$. Moreover, the $\psi_I$'s are a basis if and only if the $a_i$'s are unique. We show that the $a_I$'s are unique.

Let $J$ be any strictly increasing multi-index. Then

$$\begin{aligned}T(e_{j_1},\dots,e_{j_k}) &= \sum a_I \psi(e_{j_1},\dots,e_{j_k})\\ &= a_J,\end{aligned} \tag{4.68}$$

by property (3) of the previous theorem. Therefore, the $\psi_I$'s are a basis of $\mathcal{A}^k(V)$. □