Lecture 37

6.10 Integration on Smooth Domains

Let X be an oriented n-dimensional manifold, and let $\omega \in \Omega_c^n(X)$. We defined the integral

$$\int_X \omega, \tag{6.136}$$

but we can generalize the integral

$$\int_D \omega, \tag{6.137}$$

for some subsets $D \subseteq X$. We generalize, but only to very simple subsets called *smooth domains* (essentially manifolds-with-boundary). The prototypical smooth domain is the half plane:

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}. \tag{6.138}$$

Note that the boundary of the half plane is

$$\mathrm{Bd}\,(\mathbb{H}^n) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 = 0\}. \tag{6.139}$$

Definition 6.43. A closed subset $D \subseteq X$ is a *smooth domain* if for every point $p \in \mathrm{Bd}\,(D)$, there exists a parameterization $\phi : U \to V$ of X at p such that $\phi(U \cap \mathbb{H}^n) = V \cap D$.

Definition 6.44. The map ϕ is a *parameterization of D at p*.

Note that $\phi : U \cap \mathbb{H}^n \to V \cap D$ is a homeomorphism, so it maps boundary points to boundary points. So, it maps $U^b = U \cap \mathrm{Bd}\,(\mathbb{H}^n)$ onto $V^b = V \cap \mathrm{Bd}\,(D)$.

Let $\psi = \phi | U^b$. Then $\psi : U^b \to V^b$ is a diffeomorphism. The set U^b is an open set in $\mathbb{R}^{n-1}$, and ψ is a parameterization of the $\mathrm{Bd}\,(D)$ at p. We conclude that

$$\mathrm{Bd}\,(D) \text{ is an } (n-1)\text{-dimensional manifold.} \tag{6.140}$$

Here are some examples of how smooth domains appear in nature:

Let $f : X \to \mathbb{R}$ be a $\mathcal{C}^\infty$ map, and assume that $f^{-1}(0) \cap C_f = \phi$ (the empty set). That is, for all $p \in f^{-1}(0)$, $df_p \neq 0$.

Claim. *The set $D = \{x \in X : f(x) \leq 0\}$ is a smooth domain.*

Proof. Take $p \in \mathrm{Bd}\,(D)$, so $p = f^{-1}(0)$. Let $\phi : U \to V$ be a parameterization of X at p. Consider the map $g = f \circ \phi : U \to \mathbb{R}$. Let $q \in U$ and $p = \phi(q)$. Then

$$(dg_q) = df_p \circ (d\phi)_q. \tag{6.141}$$

We conclude that $dg_q \neq 0$.

By the canonical submersion theorem, there exists a diffeomorphism ψ such that $g \circ \psi = \pi$, where π is the canonical submersion mapping $(x_, \dots, x_n) \to x_1$. We can write simply $g \circ \psi = x_1$. Replacing $\phi = \phi_{\text{old}}$ by $\phi = \phi_{\text{new}} = \phi_{\text{old}} \circ \psi$, we get the new map $\phi : U \to V$ which is a parameterization of X at p with the property that $f \circ \phi(x_1, \dots, x_n) = x_1$. Thus, ϕ maps $\mathbb{H}^n \cap U$ onto $D \cap V$. □

We give an example of using the above claim to construct a smooth domain. Let $X = \mathbb{R}^n$, and define

$$f(x) = 1 - (x_1^2 + \cdots + x_n^2). \tag{6.142}$$

By definition,

$$f(x) \leq 0 \iff x \in B^n, \tag{6.143}$$

where $B^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ is the "unit ball." So, the unit ball B^n is a smooth domain.

We now define orientations of smooth domains. Assume that X is oriented, and let D be a smooth domain. Let $\phi : U \to V$ be a parameterization of D at p.

Definition 6.45. The map ϕ is an *oriented parameterization of* D if it is an oriented parameterization of X.

Assume that $\dim X = n > 1$. We show that you can always find an oriented parameterization.

Let $\phi : U \to V$ be a parameterization of D at p. Suppose that ϕ is *not* oriented. That is, as a diffeomorphism ϕ is orientation reversing. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be the map

$$A(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, -x_n). \tag{6.144}$$

Then A maps $\mathbb{H}^n \to \mathbb{H}^n$, and $\phi \circ A$ is orientation preserving. So, $\phi \circ A$ is an oriented parameterization of D at p.

Now, let $\phi : U \to V$ be an oriented parameterization of D at p. We define

$$U^b = U \cap \text{Bd}\,(\mathbb{H}^n), \tag{6.145}$$

$$V^b = V \cap \text{Bd}\,(D), \tag{6.146}$$

$$\psi = \phi | U^b, \tag{6.147}$$

where ψ is a parameterization of $\text{Bd}\,(D)$ at p.

We oriented $\text{Bd}\,(D)$ at p by requiring ψ to be an oriented parameterization. We need to check the following claim.

Claim. *The definition of* oriented *does not depend on the choice of parameterization.*

Proof. Let $\phi_i : U_i \to V_i,\ i = 1, 2$, be oriented parameterizations of D at p. Define

$$U_{1,2} = \phi_1^{-1}(V_1 \cap V_2), \tag{6.148}$$

$$U_{2,1} = \phi_2^{-1}(V_1 \cap V_2), \tag{6.149}$$

from which we obtain the following diagram:

$$
\begin{array}{ccc}
V_1 \cap V_2 & = & V_1 \cap V_2 \\
{\scriptstyle \phi_1}\uparrow & & {\scriptstyle \phi_2}\uparrow \\
U_{1,2} & \xrightarrow{g} & U_{2,1},
\end{array}
\tag{6.150}
$$

which defines a map g. By the properties of the other maps ϕ_1, ϕ_2, the map g is an orientation preserving diffeomorphism of $U_{1,2}$ onto $U_{2,1}$. Moreover, g maps

$$U_{1,2}^b = \mathrm{Bd}\,(\mathbb{H}^n) \cap U_{1,2} \tag{6.151}$$

onto

$$U_{2,1}^b = \mathrm{Bd}\,(\mathbb{H}^n) \cap U_{2,1}. \tag{6.152}$$

Let $h = g|U_{1,2}^b$, so $h : U_{1,2}^b \to U_{2,1}^b$. We want to show that h is orientation preserving. To show this, we write g and h in terms of coordinates.

$$g = (g_1, \dots, g_n), \quad \text{where } g_i = g_i(x_1, \dots, x_n). \tag{6.153}$$

So,

$$g \text{ maps } \mathbb{H}^n \text{ to } \mathbb{H}^n \iff \begin{cases} g_1(x_1, \dots, x_n) < 0 & \text{if } x_1 < 0, \\ g_1(x_1, \dots, x_n) > 0 & \text{if } x_1 > 0, \\ g_1(0, x_2, \dots, x_n) = 0 \end{cases} \tag{6.154}$$

These conditions imply that

$$\begin{cases} \frac{\partial}{\partial x_1} g_1(0, x_2, \dots, x_n) \geq 0, \\ \frac{\partial}{\partial x_i} g_1(0, x_2, \dots, x_n) = 0, \text{ for } i \neq 1. \end{cases} \tag{6.155}$$

The map h in coordinates is then

$$\begin{aligned} h &= h(x_2, \dots, x_n) \\ &= (g(0, x_2, \dots, x_n), \dots, g_{n-1}(0, x_2, \dots, x_n)), \end{aligned} \tag{6.156}$$

which is the statement that $h = g|\,\mathrm{Bd}\,(\mathbb{H}^n)$.

At the point $(0, x_2, \dots, x_n) \in U_{1,2}^b$,

$$Dg = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & 0 & \cdots & 0 \\ * & & & \\ \vdots & & Dh & \\ * & & & \end{bmatrix}. \tag{6.157}$$

The matrix Dg is an $n \times n$ block matrix containing the $(n-1) \times (n-1)$ matrix Dh, because

$$\frac{\partial h_i}{\partial x_j} = \frac{\partial g_i}{\partial x_j}(0, x_2, \dots, x_n), \ i, j > 1. \tag{6.158}$$

Note that

$$\det(Dg) = \frac{\partial g_1}{\partial x_1} \det(Dh). \tag{6.159}$$

We know that the l.h.s > 0 and that $\frac{\partial g_1}{\partial x_1} > 0$, so $\det(Dh) > 0$. Thus, the map $h : U^b_{1,2} \to U^b_{2,1}$ is orientation preserving.

To repeat, we showed that in the following diagram, the map h is orientation preserving:

$$\begin{array}{ccc} V_1 \cap V_2 \cap \mathrm{Bd}\,(D) & = & V_1 \cap V_2 \cap \mathrm{Bd}\,(D) \\ {\scriptstyle \psi_1}\uparrow & & {\scriptstyle \psi_2}\uparrow \\ U^b_{1,2} & \xrightarrow{h} & U^b_{2,1}. \end{array} \tag{6.160}$$

We conclude that ψ_1 is orientation preserving if and only if ψ_2 is orientation preserving. □