18.102 Introduction to Functional Analysis
Spring 2009

PROBLEM SET 5 FOR 18.102, SPRING 2009
DUE 11AM TUESDAY 17 MAR.

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You should be thinking about using Lebesgue's dominated convergence at several points below.

Problem 5.1

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be an element of $\mathcal{L}^1(\mathbb{R})$. Define

$$f_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Show that $f_L \in \mathcal{L}^1(\mathbb{R})$ and that $\int |f_L - f| \to 0$ as $L \to \infty$.

Problem 5.2

Consider a real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is locally integrable in the sense that

$$g_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & x \in \mathbb{R} \setminus [-L, L] \end{cases} \tag{5.2}$$

is Lebesgue integrable of each $L \in \mathbb{N}$.

(1) Show that for each fixed L the function

$$g_L^{(N)}(x) = \begin{cases} g_L(x) & \text{if } g_L(x) \in [-N, N] \\ N & \text{if } g_L(x) > N \\ -N & \text{if } g_L(x) < -N \end{cases} \tag{5.3}$$

is Lebesgue integrable.

(2) Show that $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$.

(3) Show that there is a sequence, h_n, of step functions such that

$$h_n(x) \to f(x) \text{ a.e. in } \mathbb{R}. \tag{5.4}$$

(4) Defining

$$h_{n,L}^{(N)} = \begin{cases} 0 & x \notin [-L, L] \\ h_n(x) & \text{if } h_n(x) \in [-N, N],\ x \in [-L, L] \\ N & \text{if } h_n(x) > N,\ x \in [-L, L] \\ -N & \text{if } h_n(x) < -N,\ x \in [-L, L] \end{cases}. \tag{5.5}$$

Show that $\int |h_{n,L}^{(N)} - g_L^{(N)}| \to 0$ as $n \to \infty$.

Problem 5.3

Show that $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space.

First working with real functions, define $\mathcal{L}^2(\mathbb{R})$ as the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which are locally integrable and such that $|f|^2$ is integrable.

(1) For such f choose h_n and define g_L, $g_L^{(N)}$ and $h_n^{(N)}$ by (5.2), (5.3) and (5.5).
(2) Show using the sequence $h_{n,L}^{(N)}$ for fixed N and L that $g_L^{(N)}$ and $(g_L^{(N)})^2$ are in $\mathcal{L}^1(\mathbb{R})$ and that $\int |(h_{n,L}^{(N)})^2 - (g_L^{(N)})^2| \to 0$ as $n \to \infty$.
(3) Show that $(g_L)^2 \in \mathcal{L}^1(\mathbb{R})$ and that $\int |(g_L^{(N)})^2 - (g_L)^2| \to 0$ as $N \to \infty$.
(4) Show that $\int |(g_L)^2 - f^2| \to 0$ as $L \to \infty$.
(5) Show that $f,\, g \in \mathcal{L}^2(\mathbb{R})$ then $fg \in \mathcal{L}^1(\mathbb{R})$ and that

$$|\int fg| \le \int |fg| \le \|f\|_{L^2}\|g\|_{L^2},\ \|f\|_{L^2}^2 = \int |f|^2. \tag{5.6}$$

(6) Use these constructions to show that $\mathcal{L}^2(\mathbb{R})$ is a linear space.
(7) Conclude that the quotient space $L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N}$, where $\mathcal{N}$ is the space of null functions, is a real Hilbert space.
(8) Extend the arguments to the case of complex-valued functions.

Problem 5.4

Consider the sequence space

$$h^{2,1} = \left\{ c : \mathbb{N} \ni j \longmapsto c_j \in \mathbb{C}; \sum_j (1+j^2)|c_j|^2 < \infty \right\}. \tag{5.7}$$

(1) Show that

$$h^{2,1} \times h^{2,1} \ni (c,d) \longmapsto \langle c, d\rangle = \sum_j (1+j^2) c_j \overline{d_j} \tag{5.8}$$

is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.

(2) Denoting the norm on this space by $\|\cdot\|_{2,1}$ and the norm on l^2 by $\|\cdot\|_2$, show that

$$h^{2,1} \subset l^2,\ \|c\|_2 \le \|c\|_{2,1}\ \forall\ c \in h^{2,1}. \tag{5.9}$$

Problem 5.5

In the separable case, prove Riesz Representation Theorem directly.

Choose an orthonormal basis $\{e_i\}$ of the separable Hilbert space H. Suppose $T : H \longrightarrow \mathbb{C}$ is a bounded linear functional. Define a sequence

$$w_i = \overline{T(e_i)},\ i \in \mathbb{N}. \tag{5.10}$$

(1) Now, recall that $|Tu| \le C\|u\|_H$ for some constant C. Show that for every finite N,

$$\sum_{j=1}^{N} |w_i|^2 \le C^2. \tag{5.11}$$

(2) Conclude that $\{w_i\} \in l^2$ and that

$$w = \sum_i w_i e_i \in H. \tag{5.12}$$

(3) Show that

$$T(u) = \langle u, w \rangle_H \ \forall \ u \in H \text{ and } \|T\| = \|w\|_H. \tag{5.13}$$

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