18.102 Introduction to Functional Analysis
Spring 2009

PROBLEM SET 2 FOR 18.102, SPRING 2009 DUE 11AM TUESDAY 24 FEB.

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I was originally going to make this problem set longer, since there is a missing Tuesday. However, I would prefer you to concentrate on getting all four of these questions really right!

1. Problem 2.1

Finish the proof of the completeness of the space B constructed in lecture on February 10. The description of that construction can be found in the notes to Lecture 3 as well as an indication of one way to proceed.

Solution. The proof could be shorter than this, I have tried to be fairly complete.

To recap. We start of with a normed space V. From this normed space we construct the new linear space $\tilde{V}$ with points the absolutely summable series in V. Then we consider the subspace $S \subset \tilde{V}$ of those absolutely summable series which converge to 0 in V. We are interested in the quotient space

$$B = \tilde{V}/S. \tag{1}$$

What we know already is that this is a normed space where the norm of $b = \{v_n\} + S$ – where $\{v_n\}$ is an absolutely summable series in V is

$$\|b\|_B = \lim_{N\to\infty} \|\sum_{n=1}^{N} v_n\|_V. \tag{2}$$

This is independent of which series is used to represent b – i.e. is the same if an element of S is added to the series.

Now, what is an absolutely summable series in B? It is a sequence $\{b_n\}$, thought of a series, with the property that

$$\sum_n \|b_n\|_B < \infty. \tag{3}$$

We have to show that it converges in B. The first task is to guess what the limit should be. The idea is that it should be a series which adds up to 'the sum of the b_n's'. Each b_n is represented by an absolutely summable series $v_k^{(n)}$ in V. So, we can just look for the usual diagonal sum of the double series and set

$$w_j = \sum_{n+k=j} v_k^{(n)}. \tag{4}$$

The problem is that this will not in generall be absolutely summable as a series in V. What we want is the estimate

$$\sum_j \|w_j\| = \sum_j \|\sum_{j=n+k} v_k^{(n)}\| < \infty. \tag{5}$$

The only way we can really estimate this is to use the triangle inequality and conclude that

$$\sum_{j=1}^{\infty}\|w_j\| \le \sum_{k,n}\|v_k^{(n)}\|_V. \tag{6}$$

Each of the sums over k on the right is finite, but we do not know that the sum over k is then finite. This is where the first suggestion comes in:-

We can *choose* the absolutely summable series $v_k^{(n)}$ representing b_n such that

$$\sum_k \|v_k^{(n)}\| \le \|b_n\|_B + 2^{-n}. \tag{7}$$

Suppose an initial choice of absolutely summable series representing b_n is u_k, so $\|b_n\| = \lim_{N\to\infty}\|\sum\limits_{k=1}^{N}u_k\|$ and $\sum\limits_k\|u_k\|_V < \infty$. Choosing M large it follows that

$$\sum_{k>M}\|u_k\|_V \le 2^{-n-1}. \tag{8}$$

With this choice of M set $v_1^{(n)} = \sum\limits_{k=1}^{M}u_k$ and $v_k^{(n)} = u_{M+k-1}$ for all $k \ge 2$. This does still represent b_n since the difference of the sums,

$$\sum_{k=1}^{N}v_k^{(n)} - \sum_{k=1}^{N}u_k = -\sum_{k=N}^{N+M-1}u_k \tag{9}$$

for all N. The sum on the right tends to 0 in V (since it is a fixed number of terms). Moreover, because of (8),

(10)

$$\sum_k\|v_k^{(n)}\|_V = \|\sum_{j=1}^{M}u_j\|_V + \sum_{k>M}\|u_k\| \le \|\sum_{j=1}^{N}u_j\| + 2\sum_{k>M}\|u_k\| \le \|\sum_{j=1}^{N}u_j\| + 2^{-n}$$

for all N. Passing to the limit as $N \to \infty$ gives (7).

Once we have chosen these 'nice' representatives of each of the b_n's if we define the w_j's by (4) then (5) means that

$$\sum_j\|w_j\|_V \le \sum_n\|b_n\|_B + \sum_n 2^{-n} < \infty \tag{11}$$

because the series b_n is absolutely summable. Thus $\{w_j\}$ defines an element of $\tilde{V}$ and hence $b \in B$.

Finally then we want to show that $\sum\limits_n b_n = b$ in B. This just means that we need to show

$$\lim_{N\to\infty}\|b - \sum_{n=1}^{N}b_n\|_B = 0. \tag{12}$$

The norm here is itself a limit – $b - \sum\limits_{n=1}^{N}b_n$ is represented by the summable series with nth term

$$w_k - \sum_{n=1}^{N}v_k^{(n)} \tag{13}$$

and the norm is then

$$\lim_{p\to\infty}\|\sum_{k=1}^{p}(w_k-\sum_{n=1}^{N}v_k^{(n)})\|_V. \tag{14}$$

Then we need to understand what happens as $N\to\infty$! Now, w_k is the diagonal sum of the $v_j^{(n)}$'s so sum over k gives the difference of the sum of the $v_j^{(n)}$ over the first p anti-diagonals minus the sum over a square with height N (in n) and width p. So, using the triangle inequality the norm of the difference can be estimated by the sum of the norms of all the 'missing terms' and then some so

$$\|\sum_{k=1}^{p}(w_k-\sum_{n=1}^{N}v_k^{(n)})\|_V\le\sum_{l+m\ge L}\|v_l^{(m)}\|_V \tag{15}$$

where $L=\min(p,N)$. This sum is finite and letting $p\to\infty$ is replaced by the sum over $l+m\ge N$. Then letting $N\to\infty$ it tends to zero by the absolute (double) summability. Thus

$$\lim_{N\to\infty}\|b-\sum_{n=1}^{N}b_n\|B=0 \tag{16}$$

which is the statelent we wanted, that $\sum_n b_n=b$. □

2. Problem 2.2

Let's consider an example of an absolutely summable sequence of step functions. For the interval $[0,1)$ (remember there is a strong preference for left-closed but right-open intervals for the moment) consider a variant of the construction of the standard Cantor subset based on 3 proceeding in steps. Thus, remove the 'central interval $[1/3,2/3)$. This leave $C_1=[0,1/3)\cup[2/3,1)$. Then remove the central interval from each of the remaining two intervals to get $C_2=[0,1/9)\cup[2/9,1/3)\cup[2/3,7/9)\cup[8/9,1)$. Carry on in this way to define successive sets $C_k\subset C_{k-1}$, each consisting of a finite union of semi-open intervals. Now, consider the *series* of step functions f_k where $f_k(x)=1$ on C_k and 0 otherwise.

(1) Check that this is an absolutely summable series.
(2) For which $x\in[0,1)$ does $\sum_k|f_k(x)|$ converge?
(3) Describe a function on $[0,1)$ which is shown to be Lebesgue integrable (as defined in Lecture 4) by the existence of this series and compute its Lebesgue integral.
(4) Is this function Riemann integrable (this is easy, not hard, if you check the definition of Riemann integrability)?
(5) Finally consider the function g which is equal to one on the union of all the subintervals of $[0,1)$ which are *removed* in the construction and zero elsewhere. Show that g is Lebesgue integrable and compute its integral.

Solution. (1) The total length of the intervals is being reduced by a factor of 1/3 each time. Thus $l(C_k)=\frac{2^k}{3^k}$. Thus the integral of f, which is non-negative, is actually

$$\int f_k=\frac{2^k}{3^k}\Longrightarrow\sum_k\int|f_k|=\sum_{k=1}^{\infty}\frac{2^k}{3^k}=2 \tag{1}$$

Thus the series is absolutely summable.

(2) Since the C_k are decreasing, $C_k \supset C_{k+1}$, only if

$$x \in E = \bigcap_k C_k \tag{2}$$

does the series $\sum_k |f_k(x)|$ diverge (to $+\infty$) otherwise it converges.

(3) The function defined as the sum of the series where it converges and zero otherwise

$$f(x) = \begin{cases} \sum_k f_k(x) & x \in \mathbb{R} \setminus E \\ 0 & x \in E \end{cases} \tag{3}$$

is integrable by definition. Its integral is by definition

$$\int f = \sum_k \int f_k = 2 \tag{4}$$

from the discussion above.

(4) The function f is not Riemann integrable since it is not bounded – and this is part of the definition. In particular for $x \in C_k \setminus C_{k+1}$, which is not an empty set, $f(x) = k$.

(5) The set F, which is the union of the intervals removed is $[0, 1) \setminus E$. Taking step functions equal to 1 on each of the intervals removed gives an absolutely summable series, since they are non-negative and the kth one has integral $1/3 \times (2/3)^{k-1}$ for $k = 1, \ldots$. This series converges to g on F so g is Lebesgue integrable and hence

$$\int g = 1. \tag{5}$$

□

3. Problem 2.3

The covering lemma for $\mathbb{R}^2$. By a rectangle we will mean a set of the form $[a_1, b_1) \times [a_2, b_2)$ in $\mathbb{R}^2$. The area of a rectangle is $(b_1 - a_1) \times (b_2 - a_2)$.

(1) We may subdivide a rectangle by subdividing either of the intervals – replacing $[a_1, b_1)$ by $[a_1, c_1) \cup [c_1, b_1)$. Show that the sum of the areas of rectangles made by any repeated subdivision is always the same as that of the original.
(2) Suppose that a finite collection of disjoint rectangles has union a rectangle (always in this same half-open sense). Show, and I really mean prove, that the sum of the areas is the area of the whole rectange. Hint:- proceed by subdivision.
(3) Now show that for any countable collection of disjoint rectangles contained in a given rectange the sum of the areas is less than or equal to that of the containing rectangle.
(4) Show that if a finite collection of rectangles has union *containing* a given rectange then the sum of the areas of the rectangles is at least as large of that of the rectangle contained in the union.
(5) Prove the extension of the preceeding result to a countable collection of rectangles with union containing a given rectangle.

Solution. (1) For subdivision of one rectangle this is clear enough. Namely we either divide the first side into two or the second side in two at an intermediate point c. After subdivision the area of the two rectanges is either

$$(1)\qquad \begin{aligned}(c-a_1)(b_2-a_2)+(b_1-c)(b_2-a_2)&=(b_1-c_1)(b_2-a_2) \text{ or}\\ (b_1-a_1)(c-a_2)+(b_1-a_1)(b_2-c)&=(b_1-c_1)(b_2-a_2).\end{aligned}$$

this shows by induction that the sum of the areas of any the rectangles made by repeated subdivision is always the same as the original.

(2) If a finite collection of disjoint rectangles has union a rectangle, say $[a_1, b_2) \times [a-2, b_2)$ then the same is true after any subdivision of any of the rectangles, by the previous result. Moreover after such subdivision the sum of the areas is always the same. Look at all the points $C_1 \subset [a_1, b_1)$ which occur as an endpoint of the first interval of one of the rectangles. Similarly let C_2 be the corresponding set of end-points of the second intervals of the rectangles. Now divide each of the rectangles repeatedly using the finite number of points in C_1 and the finite number of points in C_2. The total area remains the same and now the rectangles covering $[a_1, b_1) \times [A_2, b_2)$ are precisely the $A_i \times B_j$ where the A_i are a set of disjoint intervals covering $[a_1, b_1)$ and the B_j are a similar set covering $[a_2, b_2)$. Applying the one-dimensional result from class we see that the sum of the areas of the rectangles with first interval A_i is the product

$$(2)\qquad \text{length of } A_i \times (b_2 - a_2).$$

Then we can sum over i and use the same result again to prove what we want.

(3) For any finite collection of disjoint rectangles contained in $[a_1, b_1) \times [a_2, b_2)$ we can use the same division process to show that we can add more disjoint rectangles to cover the whole big rectangle. Thus, from the preceeding result the sum of the areas must be less than or equal to $(b_1 - a_1)(b_2 - a_2)$. For a countable collection of disjoint rectangles the sum of the areas is therefore bounded above by this constant.

(4) Let the rectangles be D_i, $i = 1, \ldots, N$ the union of which contains the rectangle D. Subdivide D_1 using all the endpoints of the intervals of D. Each of the resulting rectangles is either contained in D or is disjoint from it. Replace D_1 by the (one in fact) subrectangle contained in D. Proceeding by induction we can suppose that the first $N-k$ of the rectangles are disjoint and all contained in D and together all the rectangles cover D. Now look at the next one, D_{N-k+1}. Subdivide it using all the endpoints of the intervals for the earlier rectangles $D_1, \ldots, D_k$ and D. After subdivision of D_{N-k+1} each resulting rectangle is either contained in one of the D_j, $j \le N-k$ or is *not* contained in D. All these can be discarded and the result is to decrease k by 1 (maybe increasing N but that is okay). So, by induction we can decompose and throw away rectangles until what is left are disjoint and individually contained in D but still cover. The sum of the areas of the remaining rectangles is precisely the area of D by the previous result, so the sum of the areas must originally have been at least this large.

(5) Now, for a countable collection of rectangles covering $D = [a_1, b_1) \times [a_2, b_2)$ we proceed as in the one-dimensional case. First, we can assume that there

is a fixed upper bound C on the lengths of the sides. Make the kth rectangle a little larger by extending both the upper limits by $2^{-k}\delta$ where $\delta > 0$. The area increases, but by no more than $2C2^{-k}$. After extension the interiors of the countable collection cover the compact set $[a_1, b_1 - \delta] \times [a_2, b_1 - \delta]$. By compactness, a finite number of these open rectangles cover, and hence there semi-closed version, with the same endpoints, covers $[a_1, b_1 - \delta) \times [a_2, b_1 - \delta)$. Applying the preceeding finite result we see that

(3) $$\text{Sum of areas} + 2C\delta \geq \text{Area } D - 2C\delta.$$

Since this is true for all $\delta > 0$ the result follows.

□

I encourage you to go through the discussion of integrals of step functions – now based on rectangles instead of intervals – and see that everything we have done can be extended to the case of two dimensions. In fact if you want you can go ahead and see that everything works in $\mathbb{R}^n$!

4. Problem 2.4

(1) Show that any continuous function on $[0, 1]$ is the *uniform limit* on $[0, 1)$ of a sequence of step functions. Hint:- Reduce to the real case, divide the interval into 2^n equal pieces and define the step functions to take infimim of the continuous function on the corresponding interval. Then use uniform convergence.

(2) By using the 'telescoping trick' show that any continuous function on $[0, 1)$ can be written as the sum

(1) $$\sum_i f_j(x) \ \forall\ x \in [0, 1)$$

where the f_j are step functions and $\sum_j |f_j(x)| < \infty$ for all $x \in [0, 1)$.

(3) Conclude that any continuous function on $[0, 1]$, extended to be 0 outside this interval, is a Lebesgue integrable function on $\mathbb{R}$.

Solution. (1) Since the real and imaginary parts of a continuous function are continuous, it suffices to consider a real continous function f and then add afterwards. By the *uniform* continuity of a continuous function on a compact set, in this case $[0, 1]$, given n there exists N such that $|x - y| \leq 2^{-N} \Longrightarrow |f(x) - f(y)| \leq 2^{-n}$. So, if we divide into 2^N equal intervals, where N depends on n and we insist that it be non-decreasing as a function of n and take the step function f_n on each interval which is equal to $\min f = \inf f$ on the closure of the interval then

(2) $$|f(x) - F_n(x)| \leq 2^{-n} \ \forall\ x \in [0, 1)$$

since this even works at the endpoints. Thus $F_n \to f$ uniformly on $[0, 1)$.

(2) Now just define $f_1 = F_1$ and $f_k = F_k - F_{k-1}$ for all $k > 1$. It follows that these are step functions and that

(3) $$\sum_{k=1}^{n} = f_n.$$

Moreover, each interval for F_{n+1} is a subinterval for F_n. Since f can varying by no more than 2^{-n} on each of the intervals for F_n it follows that

$$|f_n(x)| = |F_{n+1}(x) - F_n(x)| \le 2^{-n} \ \forall \ n > 1. \tag{4}$$

Thus $\int |f_n| \le 2^{-n}$ and so the series is absolutely summable. Moreover, it actually converges everywhere on $[0,1)$ and uniformly to f by (2).

(3) Hence f is Lebesgue integrable.

(4) For some reason I did not ask you to check that

$$\int f = \int_0^1 f(x)dx \tag{5}$$

where on the right is the Riemann integral. However this follows from the fact that

$$\int f = \lim_{n\to\infty} \int F_n \tag{6}$$

and the integral of the step function is between the Riemann upper and lower sums for the corresponding partition of $[0,1]$.

□

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