18.102 Introduction to Functional Analysis

Spring 2009

Lecture 1. Tuesday, 3 Feb.

Linear spaces, metric spaces, normed spaces. Banach spaces. Examples – Euclidean spaces, continuous functions on a closed interval – $\mathcal{C}^0([0,1])$ with supremum norm. The (Riemannian) L^1 norm, outline that this is not complete on $\mathcal{C}^0([0,1])$. Brief description of l_2 – This is Hilbert space, but not explained.

What is it all for? Main aims:- To be able to carry out 'standard' constructions in (linear) functional analysis:

Abstract Hilbert space – one in each dimension

Concrete Hilbert space – Many, such as $L^2([0,1])$

Example of a theorem:- The Dirichlet problem. Let $V : [0,1] \longrightarrow \mathbb{R}$ be a real-valued function. We are interested in 'oscillating modes' on the interval; something like this arises in quantum mechanics for instance. Namely we want to know about functions $u(x)$ – twice continuously differentiable on $[0,1]$ which satisfy the differential equation

$$-\frac{d^2u}{dx^2}(x) + V(x)u(x) = \lambda u(x) \tag{1.1}$$

where λ is an 'unknown' constant – that is we want to know which λ's can occur. Well, of course all λ's can occur with $u \equiv 0$ but this is the 'trivial solution' which will always be there for such an equation. What other solutions are there? Well, there is an infinite sequence of λ's for which there is a non-trivial solution of (1.1) $\lambda_j \in \mathbb{R}$ – they are all real no non-real complex λ's can occur. For each of these there is at least one (and maybe more) 'independent' solution u_j. We can say a lot more about everything here but one main aim of this course is to get at least to this point.

Now, in fact (1.1) is just the eigenvalue equation. What we are dealing with here is an 'infinite matrix'. This is not obvious, and in fact is not a very good way of looking at things (there was such a matrix approach to quantum mechanics in the early days but it was replaced by the sort of 'operator' theory on Hilbert space that we will use here.) Still, we are in some sense dealing with infinite dimensional matrices. One of the crucial differences between infinite and finite dimensional settings is that *topology* is encountered. This is enshrined here in the notion of a *normed linear space*

Linear space:- Should I break out the axioms? It is a space V in which we can add elements and multiply by scalars with rules very similar to the basic examples of $\mathbb{R}^n$ or $\mathbb{C}^n$. Note that for us the 'scalars' are either the real numbers of the complex numbers – usually the latter. Let's be neutral and denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$ but of course consistently. Then our set V – the set of vectors with which we will deal, comes with two 'laws'. These are maps

$$+ : V \times V \longrightarrow V,\ \cdot : \mathbb{K} \times V \longrightarrow V. \tag{1.2}$$

which we denote not by $+(v,w)$ and $\cdot(s,v)$ but by $v+w$ and sv. Then we impose the *axioms of a vector space* – look them up! These are commutative group axioms for $+$ axioms for the action of $\mathbb{K}$ and the distributive law.

Our examples:

The 'trivial' case of a finite dimensional vector space.

The l^p spaces. These are spaces of sequences – the first problem set is all about them. Thus l^2 – which is a Hilbert space – consists of all the sequences

$a : \mathbb{N} \longrightarrow \mathbb{C}$ also denoted $\{a_j\}_{j=1}^{\infty}$ where $a_j = a(j)$, such that

$$\sum_{j=1}^{\infty} |a_j|^2 < \infty. \tag{1.3}$$

Seriously non-trivial examples such as $\mathrm{C}([0,1])$ the space of continuous functions on $[0,1]$ (say with complex values).

All these vector spaces carry norms, and that is what we want to talk about for most of the semester.

Definition 1. A norm on a vector space V is a function

$$\|\cdot\| : V \longrightarrow [0,\infty) \tag{1.4}$$

which satisfies the three properties:

(1) $\|v\| = 0$ iff $v = 0$.
(2) $\|tv\| = |t|\|v\|$ for all $t \in \mathbb{K}$ and $v \in V$.
(3) Triangle inequality. $\|v + w\| \le \|v\| + \|w\|$ for all $v, w \in V$.

Then show that $d(v,w) = \|v - w\|$ is a metric on V.

It follows that all the notions from 18.100 come into play – open sets, balls, closed sets, convergence of sequences, compact sets, connected sets, complete metric spaces. We will use them all!

Definition 2. A normed space which is complete with respect to the induced metric is called a *Banach* space.

I then discussed the supremum norm on $\mathcal{C}([0,1])$;

$$\|u\|_\infty = \sup_{x\in[0,1]} |u(x)|. \tag{1.5}$$

I said, but did not prove, that it this gives a Banach space.

I also disussed the L^1 norm on $\mathcal{C}([0,1])$:

$$\|u\|_{L^1} = \int_0^1 |u(x)|dx \tag{1.6}$$

and indicated why it is *not* complete. This is basically why we need to study the Lebesgue integral.

One might ask 'is the space of Riemann integrable functions on $[0,1]$ complete with respect to the L^1 norm' – indeed someone did. The answer is that $\|u\|_{L^1}$ is not even a norm on the space of Riemann integable functions. Namely it is only a 'seminorm' – the second two conditions are fine but there are non-zero functions with integral zero. So, the question is not quite precise, which is why I did not talk about it. One the other hand one can fiddle with the space (take a quotient) so that one gets a norm and then it is *not* complete. So the morally correct answer is NO, it is not a Banach space and this is true for a better (or worse) reason than it simply not being a normed space in the first place! We will get to this.

Problem set 1, Due 11AM Tuesday 10 Feb.

Full marks will be given to anyone who makes a good faith attempt to answer each question. The first four problems concern the 'little L p' spaces l^p. Note that you have the choice of doing everything for $p = 2$ or for all $1 \le p < \infty$.

Problem 1.1 Write out a proof (you can steal it from one of many places but at least write it out in your own hand) either for $p = 2$ or for each p with $1 \le p < \infty$ that

$$l^p = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty,\ a_j = a(j)\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold.

Problem 1.2 The 'tricky' part in Problem 1.1 is the triangle inequality. Suppose you knew – meaning I tell you – that for each N

$$\left(\sum_{j=1}^{N} |a_j|^p\right)^{\frac{1}{p}} \text{ is a norm on } \mathbb{C}^N$$

would that help?

Problem 1.3

Prove directly that each l^p as defined in Problem 1.1 – or just l^2 – is complete, i.e. it is a Banach space. At the risk of offending some, let me say that this means showing that each Cauchy sequence converges. The problem here is to find the limit of a given Cauchy sequence. Show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements at point N is Cauchy with respect to the norm in Problem 1.2 on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense (if you are doing $p = 2$ it is already the usual sense) and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it works.

Problem 1.4

Consider the 'unit sphere' in l^p – where if you want you can set $p = 2$. This is the set of vectors of length 1 :

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) Show that S is closed.
(2) Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e.g. by checking in Rudin).
(3) Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a 1 in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!

Problem 1.5 Show that the norm on any normed space is continuous.