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18.102 Introduction to Functional Analysis
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Lecture 21. Tuesday, April 28: Dirichlet problem on an interval

I want to do a couple of 'serious' applications of what we have done so far. There are many to choose from, and I will mention some more, but let me first consider the Diriclet problem on an interval. I will choose the interval $[0, 2\pi]$ because we looked at it before. So, what we are interested in is the problem of solving

$$-\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x) \text{ on } (0, 2\pi),\ u(0) = u(2\pi) = 0 \tag{21.1}$$

where the last part are the Dirichlet boundary conditions. I will assume that

$$V : [0, 2\pi] \longrightarrow \mathbb{R} \text{ is continuous.} \tag{21.2}$$

Now, it certainly makes sense to try to solve the equation (21.1) for say a given $f \in \mathcal{C}^0([0, 2\pi])$, looking for a solution which is twice continuously differentiable on the interval. It may not exist, depending on V but one thing we can shoot for, which has the virtue of being explicit is the following:

Proposition 28. *If $V \geq 0$ as in* (21.2) *then for each $f \in \mathcal{C}^0([0, 2\pi])$ there exists a unique twice continuously differentiable solution to* (21.1).

You will see that it is a bit hard to approach this directly – especially if you remember some ODE theory from 18.03. There are in fact various approaches to this but we want to go through L^2 theory – not surprisingly of course. How to start?

Well, we do know how to solve (21.1) if $V \equiv 0$ since we can use (Riemann) integration. Thus, ignoring the boundary conditions we can find a solution to $-d^2v/dx^2 = f$ on the interval by integrationg twice:

$$v(x) = -\int_0^x \int_0^y f(t)dtdy \text{ satifies } -d^2v/dx^2 = f \text{ on } (0, 2\pi). \tag{21.3}$$

Moroever v really is twice continuously differentiable if f is continuous. So, what has this got to do with operators? Well, we can change the order of integration in (21.3) to write v as

$$v(x) = -\int_0^x \int_t^x f(t)dydt = \int_0^{2\pi} a(x,t)f(t)dt,\ a(x,t) = (t-x)H(x-t) \tag{21.4}$$

where the Heaviside function $H(y)$ is 1 when $y \geq 0$ and 0 when $y < 0$. Thus $a(x, t)$ is actually continuous on $[0, 2\pi] \times [0, 2\pi]$ since the $t - x$ factor vanishes at the jump in $H(t - x)$. Thus v is given by applying an integral operator to f.

Before thinking more seriously about this, recall that there are also boundary conditions. Clearly, $v(0) = 0$ since we integrated from there. However, there is no particular reason why

$$v(2\pi) = \int_0^{2\pi} (t - 2\pi)f(t)dt \tag{21.5}$$

should vanish. However, we can always add to v any linear function and still satify the differential equation. Since we do not want to spoil the vanishing at $x = 0$ we can only afford to add cx but if we choose c correctly, namely consider

$$c = -\frac{1}{2\pi}\int_0^{2\pi} (t - 2\pi)f(t)dt, \text{ then } (v + cx)(2\pi) = 0. \tag{21.6}$$

So, finally the solution we want is

$$w(x) = \int_0^{2\pi} b(x,t)f(t)dt,\ b(x,t) = (t-x)H(x-t) - \frac{t-2\pi}{2\pi}x. \tag{21.7}$$

This is the unique, twice continuously differentiable, solution of $-d^2w/dx^2 = f$ in $(0, 2\pi)$ which vanishes at both end points.

Lemma 16. *The integral operator* (21.7) *extends by continuity from* $\mathcal{C}^0([0,2\pi])$ *to a compact, self-adjoint operator on* $L^2(0,2\pi)$.

Proof. Since w is given by an integral operator with a continuous real-valued kernel which is even in the sense that (check it)

$$b(t,x) = b(x,t) \tag{21.8}$$

we might as well give a more general result. □

Proposition 29. *If* $b \in \mathcal{C}^0([0,2\pi]^2)$ *then*

$$Bf(x) = \int_0^{2\pi} b(x,t)f(t)dt \tag{21.9}$$

defines a compact operator on $L^2(0,2\pi)$ *if in addition* b *satisfies*

$$\overline{b(t,x)} = b(x,t) \tag{21.10}$$

then B *is self-adjoint.*

Proof. □

Now, recall from one of the Problem sets that $u_k = c\sin(kx/2)$, $k \in \mathbb{N}$, is an orthonormal basis for $L^2(0,2\pi)$. Moreover, differentiating we find straight away that

$$-\frac{d^2u_k}{dx^2} = \frac{k^2}{4}u_k. \tag{21.11}$$

Since of course $u_k(0) = 0 = u_k(2\pi)$ as well, from the uniqueness above we conclude that

$$Bu_k = \frac{4}{k^2}u_k\ \forall\ k. \tag{21.12}$$

Thus, in this case we know the orthonormal basis of eigenfunctions for B. They are the u_k, each eigenspace is 1 dimensional and the eigenvalues are $4k^{-2}$. So, this happenstance allows us to decompose B as the square of another operator defined directly on the othornormal basis. Namely

$$Au_k = \frac{2}{k}u_k \Longrightarrow B = A^2. \tag{21.13}$$

Here again it is immediate that A is a compact self-adjoint operator on $L^2(0,2\pi)$ since its eigenvalues tend to 0. In fact we can see quite a lot more than this.

Lemma 17. *The operator* A *maps* $L^2(0,2\pi)$ *into* $\mathcal{C}^0([0,2\pi])$ *and* $Af(0) = Af(2\pi) = 0$ *for all* $f \in L^2(0,2\pi)$.

Proof. If $f \in L^2(0,2\pi)$ we may expand it in Fourier-Bessel series in terms of the u_k and find

$$f = \sum_k c_k u_k,\ (c_k) \in L^2. \tag{21.14}$$

Then of course, by definition,

$$Af = \sum_k \frac{2c_k}{k} u_k. \tag{21.15}$$

Here each u_k is a bounded continuous function, with the bound on u_k being independent of k. So in fact (21.15) converges uniformly and absolutely since it is uniformly Cauchy, for any $q > p$,

$$|\sum_{k=p}^{q} \frac{2c_k}{k} u_k| \le 2|c| \sum_{k=p}^{q} |c_k| k^{-1} \le 2|c| \|f\|_{L^2} \left(\sum_{k=p}^{q} k^{-2} \right)^{\frac{1}{2}} \tag{21.16}$$

where Cauchy-Schwartz has been used. This proves that

$$A : L^2(0,2\pi) \longrightarrow \mathcal{C}^0([0,2\pi])$$

is bounded and by the uniform convergence $u_k(0) = u_k(2\pi) = 0$ for all k implies that $Af(0) = Af(2\pi) = 0$. $\square$

So, going back to our original problem we try to solve (21.1) by moving the Vu term to the right side of the equation (don't worry about regularity yet) and hope to use the observation that

$$u = -A^2(Vu) + A^2 f \tag{21.17}$$

should satisfy the equation and boundary conditions. In fact, let's hope that $u = Av$, which has to be true if (21.17) holds with $v = -AVu + Af$, and look instead for

$$v = -AVAv + Af \Longrightarrow (\mathrm{Id} + AVA)v = Af. \tag{21.18}$$

So, we know that multiplication by V, which is real and continuous, is a bounded self-adjoint operator on $L^2(0,2\pi)$. Thus AVA is a self-adjoint compact operator so we can apply our spectral theory to $\mathrm{Id} + AVA$. It has a complete orthonormal basis and is invertible if and only if it has trivial null space.

An element of the null space would have to satisfy $u = -AVAu$. On the other hand we know that AVA is *positive* since

$$(AVAw, w) = (VAv, Av) = \int_{(0,2\pi)} V(x)|Av|^2 \ge 0 \Longrightarrow \int_{(0,2\pi)} |u|^2 = 0, \tag{21.19}$$

using the non-negativity of V. So, there can be no null space – all the eigenvalues of AVA are at least non-negative so -1 is not amongst them.

Thus $\mathrm{Id} + AVA$ is invertible on $L^2(0,2\pi)$ with inverse of the form $\mathrm{Id} + Q$, Q again compact and self-adjoint. So, to solve (21.18) we just need to take

$$v = (\mathrm{Id} + Q)Af \Longleftrightarrow v + AVAv = Af \text{ in } L^2(0,2\pi). \tag{21.20}$$

Then indeed

$$u = Av \text{ satisfies } u + A^2Vu = A^2 f. \tag{21.21}$$

In fact since $v \in L^2(0,2\pi)$ from (21.20) we already know that $u \in \mathcal{C}^0([0,2\pi])$ vanishes at the end points.

Moreover if $f \in \mathcal{C}^0([0,2\pi])$ we know that $Bf = A^2 f$ is twice continuously differentiable, since it is given by integrations – that is where B came from. Now, we know that u is L^2 satisfies $u = -A^2(Vu) + A^2 f$. Since $Vu \in L^2((0,2\pi)$ so is $A(Vu)$ and then, as seen above, $A(A(Vu)$ is continuous. So combining this with the result about $A^2 f$ we see that u itself is continuous and hence so is Vu. But then, going through the routine again

$$u = -A^2(Vu) + A^2 f \tag{21.22}$$

is the sum of two twice continuously differentiable functions. Thus is so itself. In fact from the properties of $B = A^2$ it satisifes

$$-\frac{d^2u}{dx^2} = -Vu + f \tag{21.23}$$

which is what the result claims. So, we have proved the existence part of Proposition 28. The uniqueness follows pretty much the same way. If there were two twice continuously differentiable solutions then the difference w would satisfy

$$-\frac{d^2w}{dx^2} + Vw = 0,\ w(0) = w(2\pi) = 0 \Longrightarrow w = -Bw = -A^2Vw. \tag{21.24}$$

Thus $w = A\phi$, $\phi = -AVw \in L^2(0,2\pi)$. Thus ϕ in turn satisfies $\phi = AVA\phi$ and hence is a solution of $(\operatorname{Id} + AVA)\phi = 0$ which we know has none (assuming $V \geq 0$). Since $\phi = 0$, $w = 0$.

This completes the proof of Proposition 28. To summarize what we have shown is that $\operatorname{Id} + AVA$ is an invertible bounded operator on $L^2(0,2\pi)$ (if $V \geq 0$) and then the solution to (21.1) is precisely

$$u = A(\operatorname{Id} + AVA)^{-1}Af \tag{21.25}$$

which is twice continuously differentiable and satisfies the Dirichlet conditions for each $f \in \mathcal{C}^0([0,2\pi])$.

Now, even if we do not assume that $V \geq 0$ we pretty much know what is happening.

Proposition 30. *For any $V \in \mathcal{C}^0([0,2\pi])$ real-valued, there is an orthonormal basis w_k of $L^2(0,2\pi)$ consisting of twice-continuously differentiable functions on $[0,2\pi]$, vanishing at the end-points and satisfying $-\frac{d^2w_k}{dx^2} + Vw_k = T_k w_k$ where $T_k \to \infty$ as $k \to \infty$. The equation (21.1) has a (twice continuously differentiable) solution for given $f \in \mathcal{C}^0([0,2\pi])$ if and only if*

$$T_K = 0 \Longrightarrow \int_{(0,2\pi)} f w_k = 0, \tag{21.26}$$

i.e. f is orthogonal to the null space of $\operatorname{Id} + A^2V$, which is the image under A of the null space of $\operatorname{Id} + AVA$, in $L^2(0,2\pi)$.

Problem set 10 (the last), Due 11AM Tuesday 5 May.

By now you should have become reasonably comfortable with a separable Hilbert space such as l_2. However, it is worthwhile checking once again that it is rather large – if you like, let me try to make you uncomfortable for one last time. An important result in this direction is Kuiper's theorem, which I will *not* ask you to prove[1]. However, I want you to go through the closely related result sometimes known as *Eilenberg's swindle*. Perhaps you will appreciate the little bit of trickery. First some preliminary results. Note that everything below is a closed curve in the $x \in [0,1]$ variable – you might want to identify this with a circle instead, I just did it the primitive way.

Problem P10.1 Let H be a separable, infinite dimensional Hilbert space. Show that the direct sum of two copies of H is a Hilbert space with the norm

$$H \oplus H \ni (u_1, u_2) \longmapsto (\|u_1\|_H^2 + \|u_2\|_H^2)^{\frac{1}{2}} \tag{21.27}$$

either by constructing an isometric isomorphism

$$T : H \longrightarrow H \oplus H, \text{ 1-1 and onto, } \|u\|_H = \|Tu\|_{H \oplus H} \tag{21.28}$$

or otherwise. In any case, construct a map as in (21.28).

Problem P10.2 One can repeat the preceding construction any finite number of times. Show that it can be done 'countably often' in the sense that if H is a separable, infinite dimensional, Hilbert space then

$$l_2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|_{l_2(H)}^2 = \sum_i \|u_i\|_H^2 < \infty\} \tag{21.29}$$

has a Hilbert space structure and construct an explicit isometric isomorphism from $l_2(H)$ to H.

Problem P10.3 Recall, or perhaps learn about, the winding number of a closed curve with values in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We take as given the following fact:[2] If $Q = [0,1]^N$ and $f : Q \longrightarrow \mathbb{C}^*$ is continuous then for each choice of $b \in \mathbb{C}$ satisfying $\exp(2\pi i b) = f(0)$, there exists a unique continuous function $F : Q \longrightarrow \mathbb{C}$ satisfying

$$\exp(2\pi i F(q)) = f(q), \ \forall \ q \in Q \text{ and } F(0) = b. \tag{21.30}$$

Of course, you are free to change b to $b + n$ for any $n \in \mathbb{Z}$ but then F changes to $F + n$, just shifting by the same integer.

(1) Now, suppose $c : [0,1] \longrightarrow \mathbb{C}^*$ is a closed curve – meaning it is continuous and $c(1) = c(0)$. Let $C : [0,1] \longrightarrow \mathbb{C}$ be a choice of F for $N = 1$ and $f = c$. Show that the winding number of the closed curve c may be defined *unambiguously* as

$$\operatorname{wn}(c) = C(1) - C(0) \in \mathbb{Z}. \tag{21.31}$$

(2) Show that $\operatorname{wn}(c)$ is *constant under homotopy*. That is if $c_i : [0,1] \longrightarrow \mathbb{C}^*$, $i = 1, 2$, are two closed curves so $c_i(1) = c_i(0)$, $i = 1, 2$, which are *homotopic* through closed curves in the sense that there exists $f : [0,1]^2 \longrightarrow \mathbb{C}^*$

[1] Kuiper's theorem says that for any (norm) continuous map, say from any compact metric space, $g : M \longrightarrow \mathrm{GL}(H)$ with values in the invertible operators on a separable infinite-dimensional Hilbert space there exists a continuous map, an homotopy, $h : M \times [0,1] \longrightarrow \mathrm{GL}(H)$ such that $h(m,0) = g(m)$ and $h(m,1) = \mathrm{Id}_H$ for all $m \in M$.

[2] Of course, you are free to give a proof – it is not hard.

continuous and such that $f(0,x) = c_1(x)$, $f(1,x) = c_2(x)$ for all $x \in [0,1]$ and $f(y,0) = f(y,1)$ for all $y \in [0,1]$, then $\mathrm{wn}(c_1) = \mathrm{wn}(c_2)$.

(3) Consider the closed curve $L_n : [0,1] \ni x \longmapsto e^{2\pi i x} \mathrm{Id}_{n\times n}$ of $n \times n$ matrices. Using the standard properties of the determinant, show that this curve is not homotopic to the identity through closed curves in the sense that there does not exist a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(n)$, with values in the invertible $n \times n$ matrices, such that $G(0,x) = L_n(x)$, $G(1,x) \equiv \mathrm{Id}_{n\times n}$ for all $x \in [0,1]$, $G(y,0) = G(y,1)$ for all $y \in [0,1]$.

Problem P10.4 Consider the closed curve corresponding to L_n above in the case of a separable but now infinite dimensional Hilbert space:

$$L : [0,1] \ni x \longmapsto e^{2\pi i x} \mathrm{Id}_H \in \mathrm{GL}(H) \subset \mathcal{B}(H) \tag{21.32}$$

taking values in the invertible operators on H. Show that after identifying H with $H \oplus H$ as above, there is a continuous map

$$M : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{21.33}$$

with values in the invertible operators and satisfying

$$M(0,x) = L(x),\ M(1,x)(u_1,u_2) = (e^{4\pi i x}u_1, u_2),\ M(y,0) = M(y,1),\ \forall\, x, y \in [0,1]. \tag{21.34}$$

Hint: So, think of $H \oplus H$ as being 2-vectors (u_1, u_2) with entries in H. This allows one to think of 'rotation' between the two factors. Indeed, show that

$$U(y)(u_1,u_2) = (\cos(\pi y/2)u_1 + \sin(\pi y/2)u_2, -\sin(\pi y/2)u_1 + \cos(\pi y/2)u_2) \tag{21.35}$$

defines a continuous map $[0,1] \ni y \longmapsto U(y) \in \mathrm{GL}(H \oplus H)$ such that $U(0) = \mathrm{Id}$, $U(1)(u_1,u_2) = (u_2, -u_1)$. Now, consider the 2-parameter family of maps

$$U^{-1}(y)V_2(x)U(y)V_1(x) \tag{21.36}$$

where $V_1(x)$ and $V_2(x)$ are defined on $H \oplus H$ as multiplication by $\exp(2\pi i x)$ on the first and the second component respectively, leaving the other fixed.

Problem P10.5 Using a rotation similar to the one in the preceeding problem (or otherwise) show that there is a continuous map

$$G : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{21.37}$$

such that

$$\begin{aligned} G(0,x)(u_1,u_2) &= (e^{2\pi i x}u_1, e^{-2\pi i x}u_2), \\ G(1,x)(u_1,u_2) &= (u_1,u_2),\ G(y,0) = G(y,1)\ \forall\, x,y \in [0,1]. \end{aligned} \tag{21.38}$$

Problem P10.6 Now, think about combining the various constructions above in the following way. Show that on $l_2(H)$ there is an homotopy like (21.37), $\tilde{G} : [0,1]^2 \longrightarrow \mathrm{GL}(l_2(H))$, (very like in fact) such that

$$\begin{aligned} \tilde{G}(0,x)\left\{u_k\right\}_{k=1}^{\infty} &= \left\{\exp((-1)^k 2\pi i x)u_k\right\}_{k=1}^{\infty}, \\ \tilde{G}(1,x) &= \mathrm{Id},\ \tilde{G}(y,0) = \tilde{G}(y,1)\ \forall\, x,y \in [0,1]. \end{aligned} \tag{21.39}$$

Problem P10.7: Eilenberg's swindle For an infinite dimenisonal separable Hilbert space, construct an homotopy – meaning a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(H)$ – with $G(0,x) = L(x)$ in (21.32) and $G(1,x) = \mathrm{Id}$ and of course $G(y,0) = G(y,1)$ for all $x, y \in [0,1]$.

Hint: Just put things together – of course you can rescale the interval at the end to make it all happen over $[0,1]$. First 'divide H into 2 copies of itself' and deform from L to $M(1, x)$ in (21.34). Now, 'divide the second H up into $l_2(H)$' and apply an argument just like the preceding problem to turn the identity on this factor into alternating terms multiplying by $\exp(\pm 4\pi i x)$ – starting with $-$. Now, you are on $H \oplus l_2(H)$, 'renumbering' allows you to regard this as $l_2(H)$ again and when you do so your curve has become alternate multiplication by $\exp(\pm 4\pi i x)$ (with + first). Finally then, apply the preceding problem again, to deform to the identity (always of course through closed curves). Presto, Eilenberg's swindle!

Solutions to Problem set 9

P9.1: Periodic functions

Let $\mathbb{S}$ be the circle of radius 1 in the complex plane, centered at the origin, $\mathbb{S} = \{z; |z| = 1\}$.

(1) Show that there is a 1-1 correspondence

$$\mathcal{C}^0(\mathbb{S}) = \{u : \mathbb{S} \longrightarrow \mathbb{C},\ \text{continuous}\} \longrightarrow \\ \{u : \mathbb{R} \longrightarrow \mathbb{C};\ \text{continuous and satisfying } u(x + 2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}. \tag{21.40}$$

Solution: The map $E : \mathbb{R} \ni \theta \longmapsto e^{2\pi i\theta} \in \mathbb{S}$ is continuous, surjective and 2π-periodic and the inverse image of any point of the circle is precisly of the form $\theta + 2\pi\mathbb{Z}$ for some $\theta \in \mathbb{R}$. Thus composition defines a map

$$E^* : \mathcal{C}^0(\mathbb{S}) \longrightarrow \mathcal{C}^0(\mathbb{R}),\ E^*f = f \circ E. \tag{21.41}$$

This map is a linear bijection.

(2) Show that there is a 1-1 correspondence

$$L^2(0, 2\pi) \longleftrightarrow \{u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}); u\big|_{(0,2\pi)} \in \mathcal{L}^2(0, 2\pi) \\ \text{and } u(x + 2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}/\mathcal{N}_P \tag{21.42}$$

where $\mathcal{N}_P$ is the space of null functions on $\mathbb{R}$ satisfying $u(x + 2\pi) = u(x)$ for all $x \in \mathbb{R}$.

Solution: Our original definition of $L^2(0, 2\pi)$ is as functions on $\mathbb{R}$ which are square-integrable and vanish outside $(0, 2\pi)$. Given such a function u we can define an element of the right side of (21.42) by assigning a value at 0 and then extending by periodicity

$$\tilde{u}(x) = u(x - 2n\pi),\ n \in \mathbb{Z} \tag{21.43}$$

where for each $x \in \mathbb{R}$ there is a unique integer n so that $x - 2n\pi \in [0, 2\pi)$. Null functions are mapped to null functions his way and changing the choice of value at 0 changes $\tilde{u}$ by a null function. This gives a map as in (21.42) and restriction to $(0, 2\pi)$ is a 2-sided invese.

(3) If we denote by $L^2(\mathbb{S})$ the space on the left in (21.42) show that there is a dense inclusion

$$\mathcal{C}^0(\mathbb{S}) \longrightarrow L^2(\mathbb{S}). \tag{21.44}$$

Solution: Combining the first map and the inverse of the second gives an inclusion. We know that continuous functions vanishing near the end-points of $(0, 2\pi)$ are dense in $L^2(0, 2\pi)$ so density follows.

So, the idea is that we can freely think of functions on $\mathbb{S}$ as 2π-periodic functions on $\mathbb{R}$ and conversely.

P9.2: Schrödinger's operator

Since that is what it is, or at least it is an example thereof:

$$-\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x),\ x \in \mathbb{R}, \tag{21.45}$$

(1) First we will consider the special case $V = 1$. Why not $V = 0$? – Don't try to answer this until the end!

Solution: The reason we take $V = 1$, or at least some other positive constant is that there is 1-d space of periodic solutions to $d^2u/dx^2 = 0$, namely the constants.

(2) Recall how to solve the differential equation

$$-\frac{d^2u(x)}{dx^2} + u(x) = f(x),\ x \in \mathbb{R}, \tag{21.46}$$

where $f(x) \in \mathcal{C}^0(\mathbb{S})$ is a continuous, 2π-periodic function on the line. Show that there is a unique 2π-periodic and twice continuously differentiable function, u, on $\mathbb{R}$ satisfying (21.46) and that this solution can be written in the form

$$u(x) = (Sf)(x) = \int_{0,2\pi} A(x,y)f(y) \tag{21.47}$$

where $A(x,y) \in \mathcal{C}^0(\mathbb{R}^2)$ satisfies $A(x+2\pi, y+2\pi) = A(x,y)$ for all $(x,y) \in \mathbb{R}$.

Extended hint: In case you managed to avoid a course on differential equations! First try to find a solution, igonoring the periodicity issue. To do so one can (for example, there are other ways) factorize the differential operator involved, checking that

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(\frac{dv}{dx} + v) \text{ if } v = \frac{du}{dx} - u \tag{21.48}$$

since the cross terms cancel. Then recall the idea of integrating factors to see that

$$\begin{aligned} &\frac{du}{dx} - u = e^x \frac{d\phi}{dx},\ \phi = e^{-x}u, \\ &\frac{dv}{dx} + v = e^{-x}\frac{d\psi}{dx},\ \psi = e^x v. \end{aligned} \tag{21.49}$$

Now, solve the problem by integrating twice from the origin (say) and hence get a solution to the differential equation (21.46). Write this out explicitly as a double integral, and then change the order of integration to write the solution as

$$u'(x) = \int_{0,2\pi} A'(x,y)f(y)dy \tag{21.50}$$

where A' is continuous on $\mathbb{R}\times[0,2\pi]$. Compute the difference $u'(2\pi) - u'(0)$ and $\frac{du'}{dx}(2\pi) - \frac{du'}{dx}(0)$ as integrals involving f. Now, add to u' as solution to the *homogeneous* equation, for $f = 0$, namely $c_1e^x + c_2e^{-x}$, so that the new solution to (21.46) satisfies $u(2\pi) = u(0)$ and $\frac{du}{dx}(2\pi) = \frac{du}{dx}(0)$. Now, check that u is given by an integral of the form (21.47) with A as stated.

Solution: Integrating once we find that if $v = \frac{du}{dx} - u$ then

$$v(x) = -e^{-x}\int_0^x e^s f(s)ds,\ u'(x) = e^x \int_0^x e^{-t}v(t)dt \tag{21.51}$$

gives a solution of the equation $-\frac{d^2u'}{dx^2}+u'(x)=f(x)$ so combinging these two and changing the order of integration

$$
\begin{gathered}
u'(x)=\int_0^x \tilde{A}(x,y)f(y)dy,\ \tilde{A}(x,y)=\frac{1}{2}\left(e^{y-x}-e^{x-y}\right)\\
u'(x)=\int_{(0,2\pi)} A'(x,y)f(y)dy,\ A'(x,y)=\begin{cases}\frac{1}{2}\left(e^{y-x}-e^{x-y}\right) & x\ge y\\ 0 & x\le y.\end{cases}
\end{gathered}
\tag{21.52}
$$

Here A' is continuous since $\tilde{A}$ vanishes at $x=y$ where there might otherwise be a discontinuity. This is the only solution which vanishes with its derivative at 0. If it is to extend to be periodic we need to add a solution of the homogeneous equation and arrange that

$$
u=u'+u'',\ u''=ce^x+de^{-x},\ u(0)=u(2\pi),\ \frac{du}{dx}(0)=\frac{du}{dx}(2\pi). \tag{21.53}
$$

So, computing away we see that

$$
u'(2\pi)=\int_0^{2\pi}\frac{1}{2}\left(e^{y-2\pi}-e^{2\pi-y}\right)f(y),\ \frac{du'}{dx}(2\pi)=-\int_0^{2\pi}\frac{1}{2}\left(e^{y-2\pi}+e^{2\pi-y}\right)f(y). \tag{21.54}
$$

Thus there is a unique solution to (21.53) which must satify

$$
\begin{gathered}
c(e^{2\pi}-1)+d(e^{-2\pi}-1)=-u'(2\pi),\ c(e^{2\pi}-1)-d(e^{-2\pi}-1)=-\frac{du'}{dx}(2\pi)\\
(e^{2\pi}-1)c=\frac{1}{2}\int_0^{2\pi}\left(e^{2\pi-y}\right)f(y),\ (e^{-2\pi}-1)d=-\frac{1}{2}\int_0^{2\pi}\left(e^{y-2\pi}\right)f(y).
\end{gathered}
\tag{21.55}
$$

Putting this together we get the solution in the desired form:

$$
\begin{gathered}
u(x)=\int_{(0.2\pi)} A(x,y)f(y),\ A(x,y)=A'(x,y)+\frac{1}{2}\frac{e^{2\pi-y+x}}{e^{2\pi}-1}-\frac{1}{2}\frac{e^{-2\pi+y-x}}{e^{-2\pi}-1}\Longrightarrow\\
A(x,y)=\frac{\cosh(|x-y|-\pi)}{e^{\pi}-e^{-\pi}}.
\end{gathered}
\tag{21.56}
$$

(3) Check, either directly or indirectly, that $A(y,x)=A(x,y)$ and that A is real.

Solution: Clear from (21.56).

(4) Conclude that the operator S extends by continuity to a bounded operator on $L^2(\mathbb{S})$.

Solution. We know that $\|S\|\le\sqrt{2\pi}\sup|A|$.

(5) Check, probably indirectly rather than directly, that

$$
S(e^{ikx})=(k^2+1)^{-1}e^{ikx},\ k\in\mathbb{Z}. \tag{21.57}
$$

Solution. We know that Sf is the unique solution with periodic boundary conditions and e^{ikx} satisfies the boundary conditions and the equation with $f=(k^2+1)e^{ikx}$.

(6) Conclude, either from the previous result or otherwise that S is a compact self-adjoint operator on $L^2(\mathbb{S})$.

Soluion: Self-adjointness and compactness follows from (21.57) since we know that the $e^{ikx}/\sqrt{2\pi}$ form an orthonormal basis, so the eigenvalues of S

tend to 0. (Myabe better to say it is approximable by finite rank operators by truncating the sum).

(7) Show that if $g \in \mathcal{C}^0(\mathbb{S}))$ then Sg is twice continuously differentiable. Hint: Proceed directly by differentiating the integral.

Solution: Clearly Sf is continuous. Going back to the formula in terms of $u' + u''$ we see that both terms are twice continuously differentiable.

(8) From (21.57) conclude that $S = F^2$ where F is also a compact self-adjoint operator on $L^2(\mathbb{S})$ with eigenvalues $(k^2+1)^{-\frac{1}{2}}$.

Solution: Define $F(e^{ikx}) = (k^2+1)^{-\frac{1}{2}} e^{ikx}$. Same argument as above applies to show this is compact and self-adjoint.

(9) Show that $F : L^2(\mathbb{S}) \longrightarrow \mathcal{C}^0(\mathbb{S})$.

Solution. The series for Sf

$$Sf(x) = \frac{1}{2\pi} \sum_k (2k^2+1)^{-\frac{1}{2}} (f, e^{ikx}) e^{ikx} \tag{21.58}$$

converges absolutely and uniformly, using Cauchy's inequality – for instance it is Cauchy in the supremum norm:

$$\| \sum_{|k|>p} (2k^2+1)^{-\frac{1}{2}} (f, e^{ikx}) e^{ikx} | \le \epsilon \|f\|_{L^2} \tag{21.59}$$

for p large since the sum of the squares of the eigenvalues is finite.

(10) Now, going back to the real equation (21.45), we assume that V is continuous, real-valued and 2π-periodic. Show that if u is a twice-differentiable 2π-periodic function satisfying (21.45) for a given $f \in \mathcal{C}^0(\mathbb{S})$ then

$$u + S((V-1)u) = Sf \text{ and hence } u = -F^2((V-1)u) + F^2 f \tag{21.60}$$

and hence conclude that

$$u = Fv \text{ where } v \in L^2(\mathbb{S}) \text{ satisfies } v + (F(V-1)F)v = Ff \tag{21.61}$$

where $V - 1$ is the operator defined by multiplication by $V - 1$.

Solution: If u satisfies (21.45) then

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(V(x)-1)u(x) + f(x) \tag{21.62}$$

so by the uniquenss of the solution with periodic boundary conditions, $u = -S(V-1)u + Sf$ so $u = F(-F(V-1)u + Ff)$. Thus indeed $u = Fv$ with $v = -F(V-1)u + Ff$ which means that v satisfies

$$v + F(V-1)Fv = Ff. \tag{21.63}$$

(11) Show the converse, that if $v \in L^2(\mathbb{S})$ satisfies

$$v + (F(V-1)F)v = Ff,\ f \in \mathcal{C}^0(\mathbb{S}) \tag{21.64}$$

then $u = Fv$ is 2π-periodic and twice-differentiable on $\mathbb{R}$ and satisfies (21.45).

Solution. If $v \in L^2(0, 2\pi)$ satisfies (21.64) then $u = Fv \in \mathcal{C}^0(\mathbb{S})$ satisfies $u + F^2(V-1)u = F^2 f$ and since $F^2 = S$ maps $\mathcal{C}^0(\mathbb{S})$ into twice continuously differentiable functions it follows that u satisfies (21.45).

(12) Apply the Spectral theorem to $F(V-1)F$ (including why it applies) and show that there is a sequence λ_j in $\mathbb{R}\setminus\{0\}$ with $|\lambda_j|\to 0$ such that for all $\lambda\in\mathbb{C}\setminus\{0\}$, the equation

$$\lambda v + (F(V-1)F)v = g,\ g\in L^2(\mathbb{S}) \tag{21.65}$$

has a unique solution for every $g\in L^2(\mathbb{S})$ if and only if $\lambda\neq\lambda_j$ for any j.

Solution: We know that $F(V-1)F$ is self-adjoint and compact so $L^2(0.2\pi)$ has an orthonormal basis of eigenfunctions of $-F(V-1)F$ with eigenvalues λ_j. This sequence tends to zero and (21.65), for given $\lambda\in\mathbb{C}\setminus\{0\}$, if and only if has a solution if and only if it is an isomorphism, meaning $\lambda\neq\lambda_j$ is not an eigenvalue of $-F(V-1)F$.

(13) Show that for the λ_j the solutions of

$$\lambda_j v + (F(V-1)F)v = 0, v\in L^2(\mathbb{S}), \tag{21.66}$$

are all continuous 2π-periodic functions on $\mathbb{R}$.

Solution: If v satisfies (21.66) with $\lambda_j\neq 0$ then $v=-F(V-1)F/\lambda_j\in\mathcal{C}^0(\mathbb{S})$.

(14) Show that the corresponding functions $u=Fv$ where v satisfies (21.66) are all twice continuously differentiable, 2π-periodic functions on $\mathbb{R}$ satisfying

$$-\frac{d^2u}{dx^2}+(1-s_j+s_jV(x))u(x)=0,\ s_j=1/\lambda_j. \tag{21.67}$$

Solution: Then $u=Fv$ satisfies $u=-S(V-1)u/\lambda_j$ so is twice continuously differentiable and satisfies (21.67).

(15) Conversely, show that if u is a twice continuously differentiable, 2π-periodic function satisfying

$$-\frac{d^2u}{dx^2}+(1-s+sV(x))u(x)=0,\ s\in\mathbb{C}, \tag{21.68}$$

and u is not identically 0 then $s=s_j$ for some j.

Solution: From the uniquess of periodic solutions $u=-S(V-1)u/\lambda_j$ as before.

(16) Finally, conclude that Fredholm's alternative holds for the equation (21.45)

Theorem 16. *For a given real-valued, continuous 2π-periodic function V on $\mathbb{R}$, either* (21.45) *has a unique twice continuously differentiable, 2π-periodic, solution for each f which is continuous and 2π-periodic or else there exists a finite, but positive, dimensional space of twice continuously differentiable 2π-periodic solutions to the homogeneous equation*

$$-\frac{d^2w(x)}{dx^2}+V(x)w(x)=0,\ x\in\mathbb{R}, \tag{21.69}$$

and (21.45) *has a solution if and only if $\int_{(0,2\pi)} fw=0$ for every 2π-periodic solution, w, to* (21.69).

Solution: This corresponds to the special case $\lambda_j=1$ above. If λ_j is not an eigenvalue of $-F(V-1)F$ then

$$v+F(V-1)Fv=Ff \tag{21.70}$$

has a unque solution for all f, otherwise the necessary and sufficient condition is that $(v,Ff)=0$ for all v' satisfying $v'+F(V-1)Fv'=0$. Correspondingly either

(21.45) has a unique solution for all f or the necessary and sufficient condition is that $(Fv', f) = 0$ for all $w = Fv'$ (remember that F is injetive) satisfying (21.69).

Not to be handed in, just for the enthusiastic

Check that we really can understand all the 2π periodic eigenfunctions of the Schrödinger operator using the discussion above. First of all, there was nothing sacred about the addition of 1 to $-d^2/dx^2$, we could add any positive number and get a similar result – the problem with 0 is that the constants satisfy the homogeneous equation $d^2u/dx^2 = 0$. What we have shown is that the operator

$$u \longmapsto Qu = -\frac{d^2u}{dx^2}u + Vu \tag{21.71}$$

applied to twice continuously differentiable functions has at least a left inverse unless there is a non-trivial solution of

$$-\frac{d^2u}{dx^2}u + Vu = 0. \tag{21.72}$$

Namely, the left inverse is $R = F(\mathrm{Id} + F(V-1)F)^{-1}F$. This is a compact self-adjoint operator. Show – and there is still a bit of work to do – that (twice continuously differentiable) eigenfunctions of Q, meaning solutions of $Qu = \tau u$ are precisely the non-trivial solutions of $Ru = \tau^{-1}u$.

What to do in case (21.72) does have a non-trivial solution? Show that the space of these is finite dimensional and conclude that essentially the same result holds by working on the orthocomplement in $L^2(\mathbb{S})$.