18.112 Functions of a Complex Variable
Fall 2008

Solution for 18.112 ps 6

1(Prob 1 on P193).

Solution: The partial product

$$\begin{aligned} P_n &= \prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) \\ &= \prod_{k=2}^{n}\left(1-\frac{1}{k}\right)\left(1+\frac{1}{k}\right) \\ &= \prod_{k=2}^{n}\frac{k+1}{k}\frac{k-1}{k} \\ &= \frac{n+1}{2n}, \end{aligned}$$

thus

$$\begin{aligned} \prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) &= \lim_{n\to\infty} P_n \\ &= \lim_{n\to\infty}\frac{n+1}{2n} \\ &= \frac{1}{2}. \end{aligned}$$

2(Prob 2 on P193).

Method 1. Note that

$$\begin{aligned} (1-z)P_n &= (1-z)(1+z)\cdots(1+z^{2^{n-1}}) \\ &= 1-z^{2^n}. \end{aligned}$$

Taking limit, we get

$$\begin{aligned} (1-z)P &= \lim_{n\to\infty}(1-z)P_n \\ &= \lim_{n\to\infty}(1-z^{2^n}) \\ &= 1, \end{aligned}$$

thus

$$P = \frac{1}{1-z}.$$

Method 2. You can use induction to prove

$$P_n = \sum_{k=0}^{2^n-1} z^k = \frac{1-z^{2^n}}{1-z},$$

thus

$$P = \frac{1}{1-z}.$$

Method 3. You can also use the uniqueness of 2-adic expansion to get

$$P_n = \sum_{k=0}^{2^n-1} z^k.$$

3(Prob 3 on P193).

Method 1. By *Theorem 5*, it is enough to prove

$$\sum_{n=1}^{\infty} \log\left[(1+\frac{z}{n})e^{-\frac{z}{n}}\right]$$

converges absolutely and uniformly on every compact set. Take integer M big enough such that the given compact set is bounded by $M/2$. Then

$$1 + \frac{|z|}{n} + \frac{|z|^2}{n^2} + \cdots < 2$$

for $n \geq M$. So

$$\begin{aligned}\sum_{n=M}^{\infty} \left|\log[(1+\frac{z}{n})e^{-\frac{z}{n}}]\right| &= \sum_{n=M}^{\infty} \left|[\log(1+\frac{z}{n}) - \frac{z}{n}]\right| \\ &= \sum_{n=M}^{\infty} \left|[-\frac{1}{2}(\frac{z}{n})^2 + \frac{1}{3}(\frac{z}{n})^3 - \frac{1}{4}(\frac{z}{n})^4 + \cdots]\right| \\ &\leq \sum_{n=M}^{\infty} \left|\frac{z}{n}\right|^2 \left(1 + \frac{|z|}{n} + \frac{|z|^2}{n^2} + \cdots\right) \\ &\leq \frac{1}{2}M^2 \sum_{n=M}^{\infty} \frac{1}{n^2}.\end{aligned}$$

This proves

$$\sum_{n=1}^{\infty} \log\left[(1+\frac{z}{n})e^{-\frac{z}{n}}\right]$$

converges absolutely and uniformly on any compact set.

Method 2. By *Theorem 6*, it is enough to prove

$$\sum_{n=1}^{\infty} \left| e^{-\frac{z}{n}} + \frac{z}{n}e^{-\frac{z}{n}} - 1 \right|$$

converges uniformly on $|z| < R$ for any R. This is true, since

$$\begin{aligned} \left| e^{-\frac{z}{n}} + \frac{z}{n}e^{-\frac{z}{n}} - 1 \right| &= \left| \sum_{i=2}^{\infty} \frac{(-z)^i}{n^i} \left(\frac{1}{i!} - \frac{1}{(i-1)!} \right) \right| \\ &\leq \frac{|z|^2}{n^2} \sum_{i=2}^{\infty} \left(\frac{|z|}{n} \right)^{i-2} \frac{1}{(i-2)!} \\ &\leq \frac{|z|^2}{n^2} e^{\frac{|z|}{n}} \\ &\leq \frac{R^2}{n^2} e^{\frac{R}{n}}. \end{aligned}$$