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18.112 Functions of a Complex Variable
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Lecture 21 and 22: The Prime Number Theorem

(New lecture, not in Text)

The location of prime numbers is a central question in number theory. Around 1808, Legendre offered experimental evidence that the number $\pi(x)$ of primes $< x$ behaves like $x/\log x$ for large x . Tchebychev proved (1848) the partial result that the ratio of $\pi(x)$ to $x/\log x$ for large x lies between $7/8$ and $9/8$. In 1896 Hadamard and de la Valle Poussin independently proved the Prime Number Theorem that the limit of this ratio is exactly 1. Many distinguished mathematicians (particularly Norbert Wiener) have contributed to a simplification of the proof and now (by an important device by D.J. Newman and an exposition by D. Zagier) a very short and easy proof is available.

These lectures follows Zagier's account of Newman's short proof on the prime number theorem. cf:

(1) D.J.Newman, Simple Analytic Proof of the Prime Number Theorem, Amer. Math. Monthly 87 (1980), 693-697.

(2) D.Zagier, Newman's short proof of the Prime Number Theorem. Amer. Math. Monthly 104 (1997), 705-708.

The prime number theorem states that the number $\pi(x)$ of primes which are less than x is asymptotically like $\frac{x}{\log x}$:

$$\frac{\pi(x)}{x/\log x} \longrightarrow 1 \quad \text{as } x \rightarrow \infty.$$

Through Euler's product formula (I) below (text p.213) and especially through Riemann's work, $\pi(x)$ is intimately connected to the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which by the convergence of the series in $\text{Res} > 1$ is holomorphic there.

The prime number theorem is approached by use of the functions

$$\Phi(s) = \sum_{p \text{ prime}} \frac{\log p}{p^s}, \quad \mathcal{V}(x) = \sum_{p \leq x \text{ prime}} \log p.$$

Simple properties of Φ will be used to show $\zeta(s) \neq 0$ and $\Phi(s) - \frac{1}{s-1}$ holomorphic for $\text{Res} \geq 1$. Deeper properties result from writing $\Phi(s)$ as an integral on which Cauchy's theorem for contour integration can be used. This will result in the relation $\mathcal{V}(x) \sim x$ from which the prime number theorem follows easily.

$$\text{I} \quad \frac{1}{\zeta(s)} = \prod_1^{\infty} (1 - p_n^{-s}) \quad \text{for } \text{Res} > 1.$$

Proof: For each $n (1 - p_n^{-s})^{-1} = \sum_{m=0}^{\infty} p_n^{-ms}$. Putting this into the finite product

$$\prod_1^N (1 - p_1^{-s})^{-1} \text{ we obtain } \prod_1^N (1 - p_n^{-s})^{-1} = \sum_{k=1}^{\infty} n_k^{-s}. \text{ Now let } N \rightarrow \infty.$$

$$\text{II} \quad \zeta(s) - \frac{1}{s-1} \text{ extends to a holomorphic function in } \text{Res} > 0.$$

Proof: In fact for $\text{Res} > 1$,

$$\begin{aligned} \zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{dx}{x^s} \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \end{aligned}$$

But

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| = \left| \int_n^x s \frac{dy}{y^{s+1}} \right| \leq \max_{n \leq y \leq x} \left| \frac{s}{y^{s+1}} \right| \leq \frac{s}{n^{\text{Res}+1}},$$

so the sum above converges uniformly in each half-plane $\text{Res} \geq \delta$ ($\delta > 0$).

$$\text{III} \quad \mathcal{V}(x) = O(x) \quad (\text{Sharper form proved later}).$$

Proof: Since the p in the interval $n < p \leq 2n$ divides $(2n)!$ but not $n!$ we have

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \geq \binom{2n}{n} \geq \prod_{n < p \leq 2n} p = e^{\mathcal{V}(2n) - \mathcal{V}(n)},$$

Thus

$$\mathcal{V}(2n) - \mathcal{V}(n) \leq 2n \log 2. \quad (1)$$

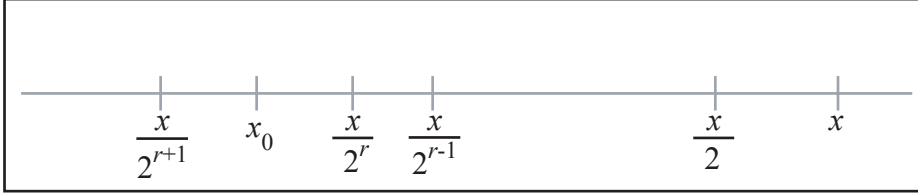
If x is arbitrary, select n with $n < \frac{x}{2} \leq n + 1$, then

$$\begin{aligned} \mathcal{V}(x) &\leq \mathcal{V}(2n + 2) \leq \mathcal{V}(n + 1) + (2n + 2) \log 2 && \text{(by (1))} \\ &= \mathcal{V}\left(\frac{x}{2} + 1\right) + (x + 2) \log 2 \\ &= \mathcal{V}\left(\frac{x}{2}\right) + \log\left(\frac{x}{2} + 1\right) + (x + 2) \log 2. \end{aligned}$$

Thus if $C > \log 2$,

$$\mathcal{V}(x) - \mathcal{V}\left(\frac{x}{2}\right) \leq Cx \quad \text{for} \quad x \geq x_0 = x_0(C). \quad (2)$$

Consider the points



Use (2) for the points right of x_0 ,

$$\begin{aligned} \mathcal{V}\left(\frac{x}{2}\right) - \mathcal{V}\left(\frac{x}{2^2}\right) &\leq C\frac{x}{2}, \\ &\vdots \\ \mathcal{V}\left(\frac{x}{2^r}\right) - \mathcal{V}\left(\frac{x}{2^{r+1}}\right) &\leq C\frac{x}{2^r}. \end{aligned}$$

Summing, we get

$$\begin{aligned} \mathcal{V}(x) - \mathcal{V}(x_0) &\leq \mathcal{V}(x) - \mathcal{V}\left(\frac{x}{2^{r+1}}\right) \\ &\leq Cx + \cdots + C\frac{x}{2^r}, \end{aligned}$$

so

$$\mathcal{V}(x) \leq 2Cx + O(1).$$

IV $\zeta(s) \neq 0$ and $\Phi(s) - \frac{1}{s-1}$ is holomorphic for $\text{Res} \geq 1$.

Proof: If $\text{Res} > 1$, part I shows that $\zeta(s) \neq 0$ and

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1} = \Phi(s) + \sum_p \frac{\log p}{p^s(p^s - 1)}. \quad (3)$$

The last sum converges for $\text{Res} > \frac{1}{2}$, so by II, $\Phi(s)$ extends meromorphically to $\text{Res} > \frac{1}{2}$ with poles only at $s = 1$ and at the zeros of $\zeta(s)$. Note that

$$\zeta(s) = 0 \quad \implies \quad \zeta(\bar{s}) = 0.$$

Let $\alpha \in \mathbb{R}$. If $s_0 = 1 + i\alpha$ is a zero of $\zeta(s)$ of order $\mu \geq 0$, then

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mu}{s - s_0} + \text{function holomorphic near } s_0.$$

So

$$\lim_{\epsilon \rightarrow 0} \epsilon \Phi(1 + \epsilon + i\alpha) = -\mu.$$

We exploit the positivity of each term in

$$\Phi(1 + \epsilon) = \sum_p \frac{\log p}{p^{1+\epsilon}}$$

for $\epsilon > 0$. It implies

$$\sum_p \frac{\log p}{p^{1+\epsilon}} \left(p^{+\frac{i\alpha}{2}} + p^{-\frac{i\alpha}{2}} \right)^2 \geq 0,$$

so

$$\Phi(1 + \epsilon + i\alpha) + \Phi(1 + \epsilon - i\alpha) + 2\Phi(1 + \epsilon) \geq 0. \quad (4)$$

By II, $s = 1$ is a simple pole of $\zeta(s)$ with residue $+1$, so

$$\lim_{\epsilon \searrow 0} \epsilon \Phi(1 + \epsilon) = 1.$$

Thus (4) implies

$$-2\mu + 2 \geq 0,$$

so

$$\mu \leq 1.$$

This is not good enough, so we try

$$\sum_p \frac{\log p}{p^{1+\epsilon}} \left(p^{+\frac{i\alpha}{2}} + p^{-\frac{i\alpha}{2}} \right)^4 \geq 0.$$

Putting

$$\lim_{\epsilon \searrow 0} \epsilon \Phi(1 + \epsilon \pm 2i\alpha) = -\nu,$$

where $\nu \geq 0$ is the order of $1 \pm 2i\alpha$ as a zero of $\zeta(s)$, the same computation gives

$$6 - 8\mu - 2\nu \geq 0,$$

which implies $\mu = 0$ since $\mu, \nu \geq 0$. Now II and (3) imply $\Phi(s) - \frac{1}{s-1}$ holomorphic for $\text{Res} \geq 1$.

$$\text{V} \quad \int_1^\infty \frac{\mathcal{V}(x) - x}{x^2} dx \text{ is convergent.}$$

Proof: The function $\mathcal{V}(x)$ is increasing with jumps $\log p$ at the points $x = p$. Thus

$$\begin{aligned} \Phi(s) &= \sum_p \frac{\log p}{p^s} \\ &= s \int_1^\infty \frac{\mathcal{V}(x)}{x^{s+1}} dx \end{aligned}$$

In fact, writing \int_1^∞ as $\sum_i \int_{p_i}^{p_{i+1}}$ this integral becomes $\sum_{i=1}^\infty \mathcal{V}(p_i) \left(\frac{1}{p_i^s} - \frac{1}{p_{i+1}^s} \right) s^{-1}$ which by $\mathcal{V}(p_{i+1}) - \mathcal{V}(p_i) = \log p_{i+1}$ reduces to $\Phi(s)$. Using the substitution $x = e^t$ we obtain

$$\Phi(s) = s \int_0^\infty e^{-st} \mathcal{V}(e^t) dt \quad \text{Res} > 1.$$

Consider now the functions

$$\begin{aligned} f(t) &= \mathcal{V}(e^t) e^{-t} - 1, \\ g(z) &= \frac{\Phi(z+1)}{z+1} - \frac{1}{z}. \end{aligned}$$

$f(t)$ is bounded by III and we have

$$\int_1^{e^T} \frac{\mathcal{V}(x) - x}{x^2} dx = \int_0^T f(t) dt. \quad (5)$$

Also, by IV,

$$\Phi(z+1) = \frac{1}{z} + h(z),$$

where h is holomorphic in $\operatorname{Re} z \geq 0$, so

$$g(z) = \frac{\Phi(z+1)}{z+1} - \frac{1}{z} = \frac{h(z) - 1}{z+1}$$

is holomorphic in $\operatorname{Re} z \geq 0$.

For $\operatorname{Re} z > 0$ we have

$$\begin{aligned} g(z) &= \int_0^\infty e^{-zt}(f(t) + 1) - \int_0^\infty e^{-zt} dt \\ &= \int_0^\infty e^{-zt} f(t) dt. \end{aligned}$$

Now we need the following theorem:

Theorem 1 (Analytic Theorem) *Let $f(t)$ ($t \geq 0$) be bounded and locally integrable and assume the function*

$$g(z) = \int_0^\infty e^{-zt} f(t) dt \quad \operatorname{Re}(z) > 0$$

extends to a holomorphic function on $\operatorname{Re}(z) \geq 0$, then

$$\lim_{T \rightarrow \infty} \int_0^T f(t) dt$$

exists and equals $g(0)$.

This will imply Part V by (5). Proof of Analytic Theorem will be given later.

VI $\mathcal{V}(x) \sim x$.

Proof: Assume that for some $\lambda > 1$ we have $\mathcal{V}(x) \geq \lambda x$ for arbitrary large x . Since $\mathcal{V}(x)$ is increasing we have for such x

$$\int_x^{\lambda x} \frac{\mathcal{V}(t) - t}{t^2} dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2} dt = \int_1^\lambda \frac{\lambda - s}{s^2} ds = \delta(\lambda) > 0.$$

On the other hand, V implies that to each $\epsilon > 0$, $\exists K$ such that

$$\left| \int_{K_1}^{K_2} \frac{\mathcal{V}(x) - x}{x^2} dx \right| < \epsilon \quad \text{for } K_1, K_2 > K.$$

Thus the λ cannot exist.

Similarly if for some $\lambda < 1$, $\mathcal{V}(x) \leq \lambda x$ for arbitrary large x , then for $t \leq x$,

$$\mathcal{V}(t) \leq \mathcal{V}(x) \leq \lambda x,$$

so

$$\int_{\lambda x}^x \frac{\mathcal{V}(t) - t}{t^2} \leq \int_{\lambda x}^x \frac{\lambda x - t}{t^2} = \int_{\lambda}^1 \frac{\lambda - s}{s^2} ds = \delta(\lambda) < 0.$$

Again this is impossible for the same reason. Thus both

$$\beta = \limsup_{x \rightarrow \infty} \frac{\mathcal{V}(x)}{x} > 1$$

and

$$\alpha = \liminf_{x \rightarrow \infty} \frac{\mathcal{V}(x)}{x} < 1$$

are impossible. Thus they must agree, i.e. $\mathcal{V}(x) \sim x$.

Proof of Prime Number Theorem:

We have

$$\mathcal{V}(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x,$$

so

$$\liminf_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} \geq \liminf_{x \rightarrow \infty} \frac{\mathcal{V}(x)}{x} = 1.$$

Secondly if $0 < \epsilon < 1$,

$$\begin{aligned} \mathcal{V}(x) &\geq \sum_{x^{1-\epsilon} \leq p \leq x} \log p \\ &\geq (1 - \epsilon) \sum_{x^{1-\epsilon} \leq p \leq x} \log x \\ &= (1 - \epsilon) \log x (\pi(x) + O(x^{1-\epsilon})) \end{aligned}$$

thus

$$\limsup_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} \leq \frac{1}{1 - \epsilon} \limsup_{x \rightarrow \infty} \frac{\mathcal{V}(x)}{x}$$

for each ϵ . Thus

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} = 1.$$

Q.E.D.

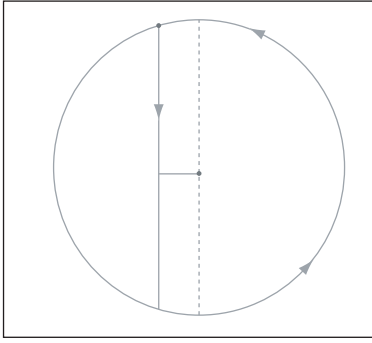
Proof of Analytic Theorem: (Newman).

Put

$$g_T(z) = \int_0^T e^{-zt} f(t) dt,$$

which is holomorphic in \mathbb{C} . We only need to show

$$\lim_{T \rightarrow \infty} g_T(0) = g(0).$$



Fix R and then take $\delta > 0$ small enough so that $g(z)$ is holomorphic on C and its interior.

By Cauchy's formula

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C (g(z) - g_T(z)) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}. \quad (6)$$

On semicircle

$$C_+ : C \cap (\operatorname{Re} z > 0)$$

integrand is bounded by $\frac{2B}{R^2}$, where

$$B = \sup_{t \geq 0} |f(t)|.$$

In fact for $\operatorname{Re} z > 0$,

$$\begin{aligned} |g(z) - g_T(z)| &= \left| \int_T^\infty f(t) e^{-zt} dt \right| \\ &\leq B \int_T^\infty |e^{-zt}| dt \\ &= \frac{B e^{-\operatorname{Re} z T}}{\operatorname{Re} z} \end{aligned}$$

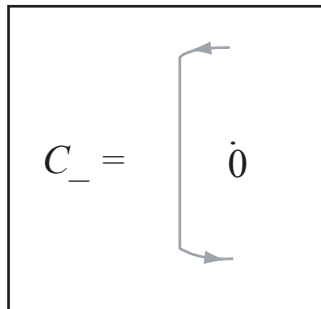
and

$$\left| e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{1}{z} \right| = e^{\operatorname{Re} z T} \cdot \frac{2\operatorname{Re} z}{R^2} \quad (z = R e^{i\theta}).$$

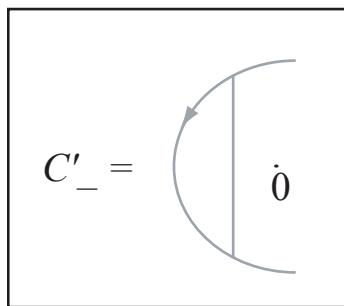
So the contribution to the integral (6) over C_+ is bounded by $\frac{B}{R}$, namely

$$\frac{Be^{-\operatorname{Re}zT}}{\operatorname{Re}z} \cdot e^{\operatorname{Re}zT} \cdot \frac{2\operatorname{Re}z}{R^2} \cdot \pi R \frac{1}{2\pi} = \frac{B}{R}.$$

Next consider the integral over



Look at $g(z)$ and $g_T(z)$ separately. For $g_T(z)$ which is entire, this contour can be replaced by



Again the integral is bounded by $\frac{B}{R}$ because

$$\begin{aligned} |g_T(z)| &= \left| \int_0^T f(t)e^{-zt} dt \right| \\ &\leq B \int_0^T |e^{-zt}| dt \\ &\leq B \int_{-\infty}^T |e^{-zt}| dt \\ &= \frac{Be^{-\operatorname{Re}zT}}{|\operatorname{Re}z|} \end{aligned}$$

and

$$\left| \left(1 + \frac{z^2}{R^2} \right) \frac{1}{z} \right|$$

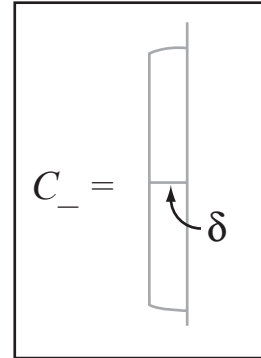
on C'_- has the same estimate as before.

There remains

$$\int_{C_-} e^{zT} \underbrace{g(z) \left(1 + \frac{z^2}{R^2} \right) \frac{1}{z}}_{\text{indep. of } T} dz.$$

On the contour, $|e^{zT}| \leq 1$ and

$$\lim_{T \rightarrow \infty} |e^{zT}| \rightarrow 0 \quad \text{for } \operatorname{Re} z < 0.$$



By dominated convergence, the integral $\rightarrow 0$ as $T \rightarrow +\infty$, δ is fixed. It follows that

$$\limsup_{T \rightarrow \infty} |g(0) - g_T(0)| \leq \frac{2B}{R}.$$

Since R is arbitrary, this proves the theorem.

Q.E.D.

Remarks: Riemann proved an explicit formula relating the zeros ρ of $\zeta(s)$ in $0 < \operatorname{Re} s < 1$ to the prime numbers. The improved version by von Mangoldt reads

$$\begin{aligned} \mathcal{V}(x) &= \sum_{p \leq x} \log p \\ &= x - \sum_{\{\rho\}} \frac{x^\rho}{\rho} + \sum_{n \geq 1} \frac{x^{-2n}}{2n} - \log 2\pi. \end{aligned}$$

He conjectured that $\operatorname{Re} \rho = \frac{1}{2}$ for all ρ . This is the famous **Riemann Hypothesis**.