Lecture 16

Chapter 4

Elliptic Operators

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4.1 Differential operators on $\mathbb{R}^n$

Let U be an open subset of $\mathbb{R}^n$ and let D_k be the differential operator,

$$\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_k}\,.$$

For every multi-index, $\alpha = \alpha_1, \ldots, \alpha_n$, we define

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}\,.$$

A differential operator of order r:

$$P : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)\,,$$

is an operator of the form

$$Pu = \sum_{|\alpha|\le r} a_\alpha D^\alpha u\,, \quad a_\alpha \in \mathcal{C}^\infty(U)\,.$$

Here $|\alpha| = \alpha_1 + \cdots \alpha_n$.

The *symbol* of P is roughly speaking its "r^{th} order part". More explicitly it is the function on $U \times \mathbb{R}^n$ defined by

$$(x,\xi) \to \sum_{|\alpha|=r} a_\alpha(x)\xi^\alpha =: p(x,\xi)\,.$$

The following property of symbols will be used to define the notion of "symbol" for differential operators on manifolds. Let $f : U \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function.

Theorem. *The operator*

$$u \in \mathcal{C}^\infty(U) \to e^{-itf}Pe^{itf}u$$

is a sum

$$\sum_{i=0}^{r} t^{r-i}P_i u \tag{4.1.1}$$

P_i being a differential operator of order i which doesn't depend on t. Moreover, P_0 is multiplication by the function

$$p_0(x) =: P(x,\xi)$$

with $\xi_i = \frac{\partial f}{\partial x_i}$, $i = 1, \ldots n$.

Proof. It suffices to check this for the operators D^α. Consider first D_k:

$$e^{-itf} D_k e^{itf} u = D_k u + t \frac{\partial f}{\partial x_k} .$$

Next consider D^α

$$\begin{aligned} e^{-itf} D^\alpha e^{itf} u &= e^{-itf} (D_1^{\alpha_1} \cdots D_n^{\alpha_n}) e^{itf} u \\ &= (e^{-itf} D_1 e^{itf})^{\alpha_1} \cdots (e^{-itf} D_n e^{itf})^{\alpha_n} u \end{aligned}$$

which is by the above

$$\big(D_1 + t\frac{\partial f}{\partial x_1}\big)^{\alpha_1} \cdots \big(D_n + t\frac{\partial f}{2 x_n}\big)^{\alpha_n}$$

and is clearly of the form (4.1.1). Moreover the t^r term of this operator is just multiplication by

$$\big(\frac{\partial}{\partial x_1} f\big)^{\alpha_1} \cdots \big(\frac{\partial}{\partial x_n}\big)^{\alpha_n} . \tag{4.1.2}$$

□

Corollary. *If P and Q are differential operators and $p(x,\xi)$ and $q(x,\xi)$ their symbols, the symbol of PQ is $p(x,\xi)\, q(x,s)$.*

Proof. Suppose P is of the order r and Q of the order s. Then

$$\begin{aligned} e^{-itf} PQ e^{itf} u &= \big(e^{-itf} P e^{itf}\big)\big(e^{-itf} Q e^{itf}\big) u \\ &= \big(p(x,\, df) t^r + \cdots\big)\big(q(x,\, df) t^s + \cdots\big) u \\ &= \big(p(x,\, df) q(x,\, df) t^{r+s} + \cdots\big) u . \end{aligned}$$

□

Given a differential operator

$$P = \sum_{|\alpha| \le r} a_\alpha D^\alpha$$

we define its *transpose* to be the operator

$$u \in \mathcal{C}^\infty(U) \to \sum_{|\alpha| \le r} D^\alpha \overline{a}_\alpha u =: P^t u .$$

Theorem. *For $u, v \in \mathcal{C}_0^\infty(U)$*

$$\langle Pu, v \rangle =: \int P u \overline{v}\, dx = \langle u, P^t \rangle .$$

Proof. By integration by parts

$$\begin{aligned} \langle D_k u, v \rangle &= \int D_k u \overline{v}\, dx = \frac{1}{\sqrt{-1}} \int \frac{\partial}{\partial x_k} u \overline{v}\, dk \\ &= -\frac{1}{\sqrt{-1}} \int u \frac{\partial}{\partial x_k} \overline{v}\, dx = \int u \overline{D_k v}\, dx \\ &= \langle u,\, d_k v \rangle . \end{aligned}$$

Thus

$$\langle D^\alpha u, v \rangle = \langle u, D^\alpha v \rangle$$

and

$$\langle a_\alpha D^\alpha u, v \rangle = \langle D^\alpha u, \overline{a}_\alpha v \rangle = \langle u, D^\alpha \overline{a}_\alpha v \rangle, .$$

□

Exercises.

If $p(x,\xi)$ is the symbol of P, $\overline{p}(x,\xi)$ is the symbol of p^t.

Ellipticity.

P is elliptic if $p(x,\xi) \notin 0$ for all $x \in U$ and $\xi \in \mathbb{R}^n - 0$.

4.2 Differential operators on manifolds.

Let U and V be open subsets of $\mathbb{R}^n$ and $\varphi : U \to V$ a diffeomorphism.
Claim. If P is a differential operator of order m on U the operator

$$u \in \mathcal{C}^\infty(V) \to (\varphi^{-1})^* P \varphi^* u$$

is a differential operator of order m on V.

Proof. $(\varphi^{-1})^* D^\alpha \varphi^* = \left((\varphi^{-1})^* D_1 \varphi^*\right)^{\alpha_1} \cdots \left((\alpha^{-1})^* D_n \varphi^*\right)^{\alpha_n}$ so it suffices to check this for D_k and for D_k this follows from the chain rule

$$D_k \varphi^* f = \sum \frac{\partial \varphi_i}{\partial x_k} \varphi^* D_i f\,.$$

□

This invariance under coordinate changes means we can define differential operators on manifolds.

Definition. Let $X = X^n$ be a real $\mathcal{C}^\infty$ manifold. An operator, $P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$, is an m^{th} order differential operator if, for every coordinate patch, $(U, x_1, \ldots, x_n)$ the restriction map

$$u \in \mathcal{C}^\infty(X) \to Pu1U$$

is given by an m^{th} order differential operator, i.e., restricted to U,

$$Pu = \sum_{|\alpha| \le m} a_\alpha D^\alpha u\,, \quad a_\alpha \in \mathcal{C}^\infty(U)\,.$$

Remark. Note that this is a non-vacuous definition. More explicitly let $(U, x_1, \ldots, x_n)$ and $(U', x'_1, \ldots, x'_n)$ be coordinate patches. Then the map

$$u \to Pu1U \cap U'$$

is a differential operator of order m in the x-coordinates if and only if it's a differential operator in the x'-coordinates.

The symbol of a differential operator

Theorem. *Let $f : X \to \mathbb{R}$ be $\mathcal{C}^\infty$ function. Then the operator*

$$u \in \mathcal{C}^\infty(X) \to e^{-itf} P e^{-itf} u$$

can be written as a sum

$$\sum_{i=0}^{m} t^{m-i} P_i$$

P_i being a differential operator of order i which doesn't depend on t.

Proof. We have to check that for every coordinate patch $(U, x_1, \ldots, x_n)$ the operator

$$u \in \mathcal{C}^\infty(X) \to e^{-itf} P e^{itf} 1U$$

has this property. This, however, follows from Theorem 4.1.

□

In particular, the operator, P_0, is a zero$^{\text{th}}$ order operator, i.e., multiplication by a $\mathcal{C}^\infty$ function, p_0.

Theorem. *There exists $\mathcal{C}^\infty$ function*

$$\sigma(P) : T^*X \to \mathbb{C}$$

not depending on f such that

$$p_0(x) = \sigma(P)(x, \xi) \tag{4.2.1}$$

with $\xi = df_x$.

Proof. It's clear that the function, $\sigma(P)$, is uniquely determined at the points, $\xi \in T^*_x$ by the property (4.2.1), so it suffices to prove the local existence of such a function on a neighborhood of x. Let $(U, x_1, \dots, x_n)$ be a coordinate patch centered at x and let $\xi_1, \dots, \xi_n$ be the cotangent coordinates on T^*U defined by

$$\xi \to \xi_1\, dx_1 + \cdots + \xi_n\, dk_n \,.$$

Then if

$$P = \sum a_\alpha D^\alpha$$

on U the function, $\sigma(P)$, is given in these coordinates by $p(x, \xi) = \sum a_\alpha(x) \xi^\alpha$. (See (4.1.2).)

□

Composition and transposes

If P and Q are differential operators of degree r and s, PQ is a differential operator of degree $r + s$, and $\sigma(PQ) = \sigma(P)\sigma(Q)$.

Let $\mathcal{F}_X$ be the sigma field of Borel subsets of X. A *measure, dx, on X* is a measure on this sigma field. A measure, dx, is *smooth* if for every coordinate patch

$$(U, x_1, \dots, x_n) \,.$$

The restriction of dx to U is of the form

$$\varphi\, dx_1 \dots dx_n \tag{4.2.2}$$

φ being a non-negative $\mathcal{C}^\infty$ function and $dx_1 \dots dx_n$ being Lebesgue measure on U. dx is *non-vanishing* if the φ in (4.2.2) is strictly positive.

Assume dx is such a measure. Given u and $v \in \mathcal{C}^\infty_0(X)$ one defines the L^2 inner product

$$\langle u, v \rangle$$

of u and v to be the integral

$$\langle u, v \rangle = \int u \overline{v}\, dx \,.$$

Theorem. *If $P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ is an m^{th} order differential operator there is a unique m^{th} order differential operator, P^t, having the property*

$$\langle Pu, v \rangle = \langle u, P^t v \rangle$$

for all $u, v \in \mathcal{C}^\infty_0(X)$.

Proof. Let's assume that the support of u is contained in a coordinate patch, $(U, x_1, \dots, x_n)$. Suppose that on U

$$P = \sum a_\alpha D^\alpha$$

and

$$dx = \varphi dx_1 \dots dx_n \,.$$

Then

$$\begin{aligned}
\langle Pu, v\rangle &= \sum_\alpha \int a_\alpha D^\alpha u \overline{v} \varphi dx_1 \dots dx_n \\
&= \sum_\alpha \int a_\alpha \varphi D^\alpha u \overline{v} dx_1 \dots dx_n \\
&= \sum \int u \overline{D^\alpha \overline{a}_\alpha \varphi v} dx_1 \dots dx_n \\
&= \sum \int u \overline{\frac{1}{\varphi} D^\alpha \varphi v} \varphi dx_1 \dots dx_n \\
&= \langle u, P^t v\rangle
\end{aligned}$$

where

$$P^t v = \frac{1}{\varphi} \sum D^\alpha \overline{a}_\alpha \varphi v \,.$$

This proves the local existence *and local uniqueness* of P^t (and hence the *global* existence of P^t!). □

Exercise.

$\sigma(P^t)(x,\xi) = \overline{\sigma(P)(x,\xi)}$.

Ellipticity.

P is elliptic if $\sigma(P)(x,\xi) \neq 0$ for all $x \in X$ and $\xi \in T_x^* - 0$.

The main goal of these notes will be to prove:

Theorem (Fredholm theorem for elliptic operators.). *If X is compact and*

$$P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

is an elliptic differential operator, the kernel of P is finite dimensional and $u \in \mathcal{C}^\infty(X)$ is in the range of P if and only if

$$\langle u, v\rangle = 0$$

for all v in the kernel of P^t.

Remark. Since P^t is also elliptic its kernel is finite dimensional.