Lecture 25

Symplectic Hodge Theory

(X^{2n}, ω) be a compact symplectic manifold. From $x \in X$ we get $\omega_x \to B_x$ a non-degenerate bilinear form on T_x^*, and so induces a non-degenerate bilinear from on $\Lambda^p(T_x^*)$.

Define $\langle , \rangle_{L^2}$ on Ω^p as follows. Take $\Omega = \omega^n/n!$, a symplectic volume form, $\alpha, \beta \in \Omega^p$

$$\langle \alpha, \beta \rangle = \int_X B_x(\alpha, \beta)\Omega = \int_X \alpha \wedge *\beta$$

Remarks:

(a) In symplectic geometry $*^2 = id$, $* = *^{-1}$.

(b) $\langle , \rangle$ is anti-symmetric on Ω^p, p odd and symmetric on Ω^p, p even.

(c) $[L^t, \delta^t] = d^t = \delta$. And $\delta^t = (d^t)^t = -d$, so $[d, L^t] = \delta$.

Consider the Laplace operator $d\delta + \delta d = dd^t + d^t d$. Now, in the symplectic world, $\Delta = 0$. We'll prove this: $\delta = [d, L^t] = dL^t - L^t d$, so $d\delta = -dL^t d$ and $\delta d = dL^t d$, so $\Delta = 0$.

So for symplectic geometry we work with the bicomplex (Ω, d, δ). We're going to use symplectic geometry to prove the Hard Lefshetz theorem for Kaehler manifolds.

Let (X^{2n}, ω) be a compact Kaehler manifold. Then we have the following operation in cohomology

$$\gamma : H^p(X, \mathbb{C}) \to H^{p+2}(X) \qquad c \mapsto [\omega] \smile c$$

Theorem (Hard Lefshetz). *γ^p is bijective.*

Question: Is Hard Lefshetz true for compact symplectic manifolds. If not, when is it true.

Define $[L^t, L] = A$, by Kaehler-Weil says that $A\alpha = (n-p)\alpha$.

Lemma. $[A, L^t] = 2L^t$.

Proof. $AL^t\alpha - L^t A\alpha = (n-(p-2))L^t\alpha - (n-p)L^t\alpha = 2L^t a$

Lemma. $[A, L] = -2L$.

There is another place in the world where you encounter these: Lie Groups.

Lie Groups

Take $G = SL(2, \mathbb{R})$, then consider the lie algebra $\mathbf{g} = sl(2, \mathbb{R})$.

This is the algebra $\{A \in M_{22}(\mathbb{R}), tr\ A = 0\}$. Generated by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Check that $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$, and $sl(2, \mathbb{R}) = span\{X, Y, Z\}$, and the above describes the Lie Algebra structure.

$\rho : \mathbf{g} \to End(\Omega)$ be given by $X \mapsto L^t$, $Y \mapsto L$ and $H \mapsto A$ is a representation of the Lie algebra $\mathbf{g}$ on Ω. So Ω is a $\mathbf{g}$-module.

Lemma. *Ω_{harm} is a $\mathbf{g}$-module of Ω.*

Proof. First note that $Ld = dL$, i.e. $dL\alpha = d(\omega \wedge \alpha) = \omega \wedge d\alpha = Ld\alpha$. Taking transposes we get $L^t\delta = \delta L^t$. Then take $\alpha \in \Omega_{harm}$. We already know that $[d, L^t] = \delta$, so $dL^t\alpha - L^t d\alpha = \delta\alpha$, which implies that $dL^t\alpha = 0$. Similarly $dL\alpha, \delta L\alpha = 0$, so $L\alpha, L^t\alpha$ are in Ω_{harm}.

So since $A = [L, L^t]$, $A\alpha \in \Omega_{harm}$ and Ω is a g-module. ∎

Note that Ω_{harm} is not finite dimensional. So these representations are not necessarily easy to deal with.

Definition. Let V be a $\mathbf{g}$-module. V is of **finite H-type** if

$$V = \bigoplus_{i=1}^{N} V_i$$

and $H = \lambda_i Id$ on V_i.

In other words, H is in diagonal form with respect to this decomposition.

Example. $\Omega = \bigoplus_{p=0}^{2n} \Omega^p$, $H = (n-p)Id$ on Ω^p and $\Omega_{harm} = \bigoplus_{p=0}^{2n} \Omega^p_{harm}$, $H = (n-p)Id$ on Ω^p_{harm}.

Theorem. *If V is a $\mathbf{g}$-module of finite type, then every sub and quotient module is of finite type.*

Proof. $V = \bigoplus_{i=1}^{N} V_i$, $H = \lambda_i Id$ on V_i. Let $\pi_i : V \to V_i$ be a projection onto V_i. Check that

$$\pi_i = \frac{1}{\prod(\lambda_i - \lambda_j)} \prod_{j \neq i} (H - \lambda_j)$$

i.e., $\pi_i v = v$ on v_i. So π_i takes sub/quotient objects onto themselves.