Lecture 36

Let G be an n-torus, and $\alpha_1, \dots, \alpha_d \in \mathbb{Z}^*_G$. Define a Hamiltonian action τ of G on $\mathbb{C}^d$ as follows. First we have

$$L : \mathbb{R}^d \to \mathfrak{g}^* \qquad L(t) = \sum t_i \alpha_i$$

and

$$\gamma : \mathbb{C}^d \to \mathbb{R}^d \qquad \gamma(z) = (|z_1|^2, \dots, |z_d|^2)$$

then $\Phi = L \circ \gamma$ is the moment map of τ. As before, we're interested in the regular values of Φ.

Define $\Delta_a = L^{-1}(a) \cap \mathbb{R}^d_+$ a convex polytope.

Theorem (1). *a is a regular value if Δ_a is a simple n-dimensional.*

For a regular call $Z_a = \Phi^{-1}(a)$. Assume G acts freely on Z_a. we have $M_a = Z_a/G$.

$$\begin{tikzcd} Z_a \arrow[r, "i"] \arrow[d, "\psi"'] & \mathbb{C}^d \\ M_a & \end{tikzcd}$$

$\psi : M_a \to \mathbb{R}^d$ and $\psi \circ \pi = \gamma \circ i$.

$Z_a = \gamma^{-1}(\Delta_a)$ implies that $\psi(M_a) = \Delta_a$.

Definition. Δ_a is called the **moment polytope**

For $\xi \in \mathbb{R}^d$, let $f = \langle \psi, \xi \rangle$ and $\pi^* f = i^* f_0$ where

$$f_0(z) = \sum_{i=1}^{d} \xi_j |z_j|^2$$

Theorem (2). *Suppose that for all adjacent v, v' of Δ_a we have $\langle v - v', \xi \rangle \neq 0$. Then*

(a) f is Morse

(b) ψ maps $Crit(f)$ bijectively onto $Vert(\Delta_a)$.

(c) For $q \in Crit(f)$ $\mathrm{ind}_q = \mathrm{ind}_\xi\, v$ where $v = \psi(a)$ and the index $ind_v \xi$ is given by

$$ind_v xi = \{v_k \mid \langle v_k - v, \xi \rangle < 0\}$$

where the v_k's are vertices adjacent to v.

Recall:
$I \subseteq \{1, \dots, d\}$ then $t \in \mathbb{R}^d_I$ if and only if $t_i \neq 0$ if and only if $i \in I$. For $\Delta = \Delta_a$

$$\mathcal{I}_\Delta = \{I, \mathbb{R}^d_I \cap \Delta \neq \emptyset\}$$

For $I \in \mathcal{I}_\Delta$, $\Delta_I = \mathbb{R}^d_I \cap \Delta =$ faces of the polytope Δ. Recall also that there is a partial ordering $I_1 \leq I_2$ if and only if $I_1 \subseteq I_2$.

For I minimal $\Delta_I = \{v_I\}$

Theorem. *a is a regular value if and only if for every vertex v_I of Δ_a, $\alpha_i, i \in I$ form a basis of $\mathfrak{g}^*$.*

Let $v_I \in Vert(\Delta_a)$. Relabel $I = (1, 2, \dots, n)$ so that $\alpha_1, \dots, \alpha_n$ are a basis for $\mathfrak{g}^*$, $a = \sum_{i=1}^n a_i \alpha_i$, $L(v_I) = a$. $v_I = (a_1, \dots, a_n, 0, \dots, 0)$ and for $k > n$,

$$\alpha_k = \sum a_{k,i} \alpha_i$$

Rewrite

$$L(t) = \sum_{i=1}^{n} \left(t_i - \sum_{k>n} a_{k,i} t_k \right) \alpha_i = \sum a_i \alpha_i = a$$

From this we conclude that Δ_a is defined by

$$(I) \begin{cases} t_i = a_i = \sum a_{k,i} t_k \\ t_1, \dots, t_d \geq 0 \end{cases} .$$

We see immediately tat Δ_a is m-dimensional, $m = d - n$. The edges of Δ_a at v_I lie along the rays $v_I + s e_k$, $k = n+1, \dots, d$ for $s \geq 0$.

Exercise Check that $e_k = (-a_{k,1}, \dots, a_{k,n}, 0, \dots, 1, \dots, 0)$ where the 1 is in the kth slot.

The conclusion is that Δ_a is simple at v_I so Δ_a is simple.

Let $v = v_I$ be a vertex of Δ_a. Write

$$\mathcal{O}_v = \{t \in \Delta_a, t_i > 0 \text{ if } i \in\} = \bigcup J \geq I \Delta_I.$$

Consider $\gamma^{-1}(\mathcal{O}_v)$. These are open G-invariant sets in Z_a

Take $\mathcal{U}_v = \pi(\gamma^{-1}(\mathcal{O}_v))$ an open cover of M_a. Let $f : M_a \to \mathbb{R}$. What does f look like on $f \mid_{\mathcal{U}_v}$. Take $I = (1, \ldots, n)$ by relabeling. Then

$$a = \sum_{i=1}^{n} a_i \alpha_i \qquad v_I = (a_1, \ldots, a_n, 0, \ldots, 0)$$

then

$$z \in \gamma^{-1}(v_I) \iff \begin{cases} |z_i|^2 = a_i & i = 1, \ldots, n \\ z_k = 0 & k > n \end{cases}$$

Proposition. *$\gamma^{-1}(v_I)$ is a single G-orbit.*

Proof. $\dim \gamma^{-1}(v) = n$, $\dim G = n$ and G acts freely on $\gamma^{-1}(v)$.

More generally, $z \in Z_a$ if and only if $\gamma(z) \in \Delta_a$. Hence by (I) $\mathcal{O}_v$ is defined by

$$|z_i|^2 = a_i - \sum a_{k,i} |z_k|^2$$

and $z_i \neq 0$, $i = 1, \ldots, n$.

Take $f_0 = \sum \xi_j |z_j|^2$ then (*)

$$i^* f_0 = c + \sum_{k>n} \left(\xi_k - \sum a_{k,i} \xi_i \right) |z_k|^2$$

$$= c + \sum_{k>n} \langle e_k, \xi \rangle |z_k|^2 = \pi^* f$$

where e_k is defined as before. □

Proof of Theorem 2. From $(*)$ the only critical point of f on $\mathcal{U}_v$ is $a = \pi(\gamma^{-1}(v))$.(Recall $\gamma^{-1}(v)$ is a single G-orbit).

Moreover $\psi(a) = v_I$. Finally if $p \in \gamma^{-1}(v)$, then

$$(d\pi)_p^* (d^2 f_a) = \sum_{k>n} \langle e_k, \xi \rangle |z_k|^2 = \sum_{k>n} \langle e_k, \xi \rangle (x_k^2 + y_k^2)$$

It follows that $(d^2 f_a)$ is (....), and the index is $2\mathrm{ind}_\xi v$. □

Also a consequence

$$H^{2k+1}(M_a) = 0$$

so

$$b_k = \dim H^{2k}(M_a) = \#\{Vert(\Delta_a), \mathrm{ind}_\xi v = k\}$$

and $b_k = \#\{\mathrm{ind}_x iv = v\}$ doesn't depend on ξ. If f_k is the number of k-dimensional faces of Δ_a for $k = 0, \ldots, m$ then

$$f_{m-k} = \binom{m}{k} b_0 + \binom{m-1}{k-1} b_1 + \cdots + b_k$$

Exercise Prove this.

Let Δ be a simple m-dimensional convex polytope and f_k be the number of k-dimensional faces of Δ. Define $b_0, \ldots, b_n$ by the solutions to the equations

$$f_{m-k} = \binom{m}{k} b_0 + \ldots b_k$$

Then

Theorem (McMullen, Stanley). *(a) The b_ks are integers.*

(b) $b_{m-k} = b_k$

(c) $b_0 \leq b_1 \leq \cdots \leq b_k$ *where* $k = [\frac{m}{2}]$.

Proof. Exhibit Δ as the moment polytope of a toric variety of M.

(a) The b_ks are Betti numbers of M (so integers)

(b) Poincare duality

(c) Hard Lefschetz.