MEASURE AND INTEGRATION: LECTURE 22

Fundamental theorem of calculus. Let $f\colon [a,b] \to \mathbb{R}$ be continuous and let $F(x) = \int_a^x f(t)\ dt$ ($a \leq x \leq b$). Then F is differentiable and $F'(x) = f(x)$. But what if $f \in L^1$ only? Claim: F is continuous if $f \in L^1$. Proof:

$$\begin{aligned} F(x) &= \int_{[a,b]} f(t)\chi_{[a,x]}\ dt \\ \lim_{x\to x_0} F(x) &= \int_{[a,b]} \lim_{x\to x_0} f(t)\chi_{[a,x]}\ dt \quad \text{by LDCT} \\ &= \int_{[a,b]} f(t)\chi_{[a,x_0]}\ dt = F(x_0). \end{aligned}$$

Lebesgue's theorem. If $f \in L^1$, then F is differentiable a.e. and $F'(x) = f(x)$ a.e.

Other half of FTOC. If given f differentiable a.e. (so f' exists), and let $f \in L^1$. Then is it true that

$$f(x) - f(a) = \int_a^x f'(t)\ dt?$$

This is not necessarily true. It is true is f is absolutely continuous on $[a,b]$ or if f is everywhere differentiable and $f' \in L^1$.

A function $f\colon [a,b] \to \mathbb{R}$ is *absolutely continuous* if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^n |f(\beta_i) - f(\alpha_i)| < \epsilon$$

for any n and collection of disjoint segments $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ in $[a,b]$ such that $\sum_{i=1}^n (\beta_i = \alpha_i) < \delta$. Absolute continuity is stronger than continuity: continuity is when $n = 1$.

Theorem 0.1. *If f is absolutely continuous, then f is differentiable a.e. and $f' \in L^1$. In particular, the second half of the FTOC holds.*

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Vitali covering theorem.

Theorem 0.2. *Let $E \subset \mathbb{R}^n$ be a bounded set, and suppose we have a collection of balls $\mathcal{F}$ containing E such that every point of E is at the center of some ball (there may be several balls at each point). Then there exist balls $B_1, B_2, \ldots \in \mathcal{F}$ so that*

1. *$B_1, B_2, \ldots$ are disjoint, and*
2. *$E \subset \bigcup_{\alpha \geq 1} 3B_\alpha$.*

Proof. Assume an upper bound to the radius of the balls (otherwise, since E is bounded, it is covered by a single ball.) Use the following selection procedure. Let $B_1, \ldots, B_{k-1}$ be selected, and define

$$d_k = \sup\{\operatorname{rad} B \mid B \in \mathcal{F}, B \cap \bigcup_{j<k} B_j = \emptyset\},$$

the largest radius of balls disjoint from the union of previously selected balls. First step is just the choosing the ball with largest radius. If no disjoint balls remain, then stop. Otherwise, choose $B_k \in \mathcal{F}$ disjoint from $B_1, \ldots, B_{k-1}$ such that $\operatorname{rad} B_k > d_k/2$. This gives a finite or countably infinite collection of disjoint balls, so (1) is satisfied. To prove (2), choose any $x \in E$, and we NTS that $x \in 3B_\alpha$ for some α. From the assumption, $x \in B$ (at center) for some B. Let $\rho = \operatorname{rad} B$. Claim: B must hit some B_k in our choice. If not, the selection process never terminates, so that $d_k > \rho$ for all k. But this implies that the collection of disjoint balls all with radius $\rho/2$, and this contradicts boundedness.

Choose smallest k so that $B \cap B_k \neq \emptyset$, so $B \cap B_j = \emptyset$ for all $j < k$. Then $\rho \leq d_k < 2 \operatorname{rad} B_k$. Choose $y \in B \cap B_k$ and let z be the center of B_α. Then

$$\begin{aligned} |x - z| &\leq |x - y| + |y - z| \\ &< \rho + \operatorname{rad} B_k \\ &< 3 \operatorname{rad} B_k, \end{aligned}$$

and thus $x \in 3B_k$. □

The maximal function. Given $f \in L^1(\mathbb{R}^n)$, the *maximal function* of f is

$$Mf(x) = \sup_{0<f<\infty} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y)| \; dy.$$

Let us show that Mf is measurable. Let $t < Mf(x)$, then there exists $0 < r < \infty$ such that

$$t < \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y)| \; dy.$$

Choose any $r' > r$ so that

$$t < \frac{1}{\lambda(B(x, r'))} \int_{B(x,r)} |f(y)| \; dy.$$

Note that if we have x' such that $|x - x'| \leq r' - r$, then $B(x, r) \subset B(x', r')$. Thus,

$$\begin{aligned} t &< \frac{1}{\lambda(B(x, r'))} \int_{B(x',r')} |f(y)| \; dy \\ &\leq \frac{1}{\lambda(B(x', r'))} \int_{B(x',r')} |f(y)| \; dy. \\ &\leq Mf(x'). \end{aligned}$$

We've shown that if $Mf(x) > t$, then for x' close to x, $Mf(x') > t$. Hence, Mf is lower semicontinuous and thus Borel measurable.

Recall that if $g \in L^1$, then

$$\|g\|_1 = \int |g| \geq \int_{\mu(\{x \mid |g(x)| \geq t\})} |g| \geq t\mu(\{x \mid |g(x)| \geq t\}).$$

Thus, Chebyshev's inequality,

$$\mu(\{x \mid |g(x)| \geq t\}) \leq \frac{\|g\|_1}{t}$$

holds. If g satisfies $\mu(\{x \mid |g(x)| > t\}) < c/t$, this does not imply that $g \in L^1$. For example, $g(x) = |x|^{-n}$ on $\mathbb{R}^n$. This condition is known as *weak* $(1, 1)$.

Theorem 0.3. *If* $f \in L^1(\mathbb{R}^n)$*, then*

$$\lambda(\{x \mid Mf(x) > t\}) \leq \frac{3^n \|f\|_1}{t} \quad \text{for all } 0 < t < \infty,$$

i.e., Mf *is weak* $(1, 1)$.

Remark. Mf is not L^1 unless $f = 0$.

Proof of remark. Choose $a > 0$ and $|x| > a$. Then

$$\begin{aligned} Mf &\geq \frac{1}{B(x, 2|x|)} \int_{B(x,2|x|)} |f(y)| \; dy \\ &\geq \frac{1}{B(0, 2|x|)} \int_{B(0,a)} |f(y)| \; dy \\ &= \frac{C}{x^n} \int_{B(0,a)} |f(y)| \; dy. \end{aligned}$$

If $Mf(x)$ is not integrable, then

$$\int_{B(0,a)} |f(y)| \ dy = 0 \Rightarrow f = 0$$

since $1/x^n$ is not integrable when x is large. □

Proof of theorem. Let $E = \{x \mid Mf(x) > t\}$. We NTS that

$$\lambda(E) \leq 3^n \|f\|_1 / t.$$

If we show that

$$\lambda(E \cap b(0,k)) \leq \frac{3^n \|f\|_1}{t}$$

for any k, then by taking limits (i.e., increasing sequence of sets), this will show that

$$\lambda(E) \leq 3^n \|f\|_1 / t.$$

So, without loss of generality, assume E is bounded.

Given $x \in E$,

$$\sup_{0<r<\infty} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y)| \ dy > t,$$

so there exists r such that

$$\sup_{0<r<\infty} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y)| \ dy > t,$$

i.e., for any $x \in E$, there exists r such that

$$\lambda(B(x,r)) < \frac{1}{t} \int_{B(x,r)} |f(y)| \ dy. \tag{0.1}$$

Let $\mathcal{F}$ be the collection of all balls centered at E satisfying (0.1). Then all the hypotheses of the Vitali covering theorem are satisfied (E bounded and every point is at the center of some ball in $\mathcal{F}$). Thus, there exists a sequence of balls $B_1, B_2, \ldots \in \mathcal{F}$ such that

(1) $B_1, B_2, \ldots$ are disjoint, and
(2) $E \subset \bigcup_{k \geq 1} 3B_k$.

Thus,

$$\begin{aligned}
\lambda(E) &\leq \sum_{k\geq 1} \lambda(3B_k) = \sum_{k\geq 1} 3^n \lambda(B_k) \\
&< \sum_{k\geq 1} 3^n \frac{1}{t} \int_{B_k} |f(y)| \ dy \\
&= 3^n \frac{1}{t} \int_{\cup_{k\geq 1} B_k} |f(y)| \ dy \\
&\leq 3^n \frac{1}{t} \int_{\mathbb{R}^n} |f(y)| \ dy.
\end{aligned}$$

□