Solving the Laplace equation in $\mathbb{R}^2$: The Dirichlet problem

In this lecture we will study something called the Dirichlet problem for discs in $\mathbb{R}^2$. Given a continuous function f on the boundary of a disc B_r we will try to construct a harmonic function u on the entire disc so that f and u agree on ∂B_r. This turns out to be a lot easier in polar coordinates.

1 The laplacian in polar coordinates.

The first problem is to calculate the laplacian in polar coordinates. If $u : \mathbb{R}^2 \to \mathbb{R}$, then

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{1}$$

This is a straightforward consequence of the chain rule, but the calculation is rather tedious. Now look for harmonic functions of the form $u(r,\theta) = g(r)h(\theta)$. Plug this into (1) and

$$h\frac{\partial^2 g}{\partial r^2} + \frac{h}{r}\frac{\partial g}{\partial r} + \frac{g}{r^2}\frac{\partial^2 h}{\partial \theta^2} = 0. \tag{2}$$

Multiplying by $\frac{r^2}{u}$ we get

$$\frac{r^2}{g}\frac{\partial^2 g}{\partial r^2} + \frac{r}{g}\frac{\partial g}{\partial r} + \frac{1}{h}\frac{\partial^2 h}{\partial \theta^2} = 0. \tag{3}$$

The first two terms depend only on radius, and the last term depends only on θ, so both must be constant. We have

$$\frac{h''}{h} = c, \tag{4}$$

and

$$\frac{r^2 g''}{g} + \frac{r g'}{g} = -c. \tag{5}$$

We can solve these separately. From (4) we get

$$h = \begin{cases} a e^{(\sqrt{c})\theta} + b e^{-(\sqrt{c})\theta} & \text{if } c > 0 \\ a + b\theta & \text{if } c = 0, \text{ or} \\ a\cos(\sqrt{-c}\theta) + b\sin(\sqrt{-c}\theta) & \text{if } c < 0, \end{cases}$$

but h has to be periodic with period 2π. This immediately excludes the possibility that $c > 0$, and if $c = 0$ we need h to be constant. It also restricts the possibilities if $c < 0$, when we need $c = -k^2$ for some integer k. Our solutions are h constant or $h = a\cos k\theta + b\sin k\theta$ for some integer k.

Solving (5) is a little trickier. First we'll deal with the case $c = 0$. Then we get

$$rg'' + g' = 0$$

This is a first order equation in g' which we can solve to get $g' = \frac{a_1}{r}$, so $g = a_0 + a_1 \log r$. In order for g to be well defined at the origin we need $a_1 = 0$, so if $c = 0$ then the only solutions are constants.

Now deal with $c \neq 0$. From above we need $c = -k^2$. We'll try for solutions of the form $g = r^m$. Plugging this into (5) we get

$$r^m(m^2 - k^2) = 0, \tag{6}$$

so we need $m = \pm k$, and $g(r) = a_0 r^k + a_1 r^{-k}$. Once again we need $a_1 = 0$ so that g is well defined at the origin. Together with our solution for h we have u constant or

$$u(r,\theta) = r^k(a\cos k\theta + b\sin k\theta) \tag{7}$$

for some integer k. Since the sum of any collection of harmonic functions is still harmonic we also get solutions of the form

$$v = \sum_{k=0}^{\infty} r^k(a_k \cos k\theta + b_k \sin k\theta). \tag{8}$$

We have found lots of harmonic functions on $\mathbb{R}^2$, and we can now solve the Dirichlet problem fairly easily. Take a continuous function f on the boundary of a disc B_R. We can expand f as a fourier series, so

$$f = \sum_{k=0}^{\infty}(c_k \cos k\theta + d_k \sin k\theta) \tag{9}$$

Define $v = \sum_{k=0}^{\infty} r^k(R^{-k}c_k \cos k\theta + R^{-k}d_k \sin k\theta)$. Then v is harmonic on B_R, and $v = f$ on the boundary as required. It turns out that all harmonic equations on the disc are of this form. This follows immediately from the fact that two harmonic functions which agree on the boundary agree over the entire set.