Lecture Seven: Consequences of Cacciopolli

1 Consequences of Cacciopolli

In this lecture we continue to generalise some of our results about the Laplacian to uniformly elliptic second order operators. We start by stating two results without proof.

Proposition 1.1 *Let B_{2r} be a ball in $\mathbb{R}^n$, and let L be a uniformly elliptic second order operator with*

$$Lf = \nabla \cdot A\nabla f,$$

and $\lambda|\mathbf{v}|^2 \leq \mathbf{v} \cdot A\mathbf{v} \leq \Lambda|\mathbf{v}|^2$. There are positive constants c, and d depending only on the dimension and the ratio $\frac{\lambda}{\Lambda}$ such that

$$\int_{B_{2r}} u^2 \geq (1+c)\int_{B_r} u^2 \tag{1}$$

and

$$\int_{B_{2r}} |\nabla u|^2 \geq (1+d)\int_{B_r} |\nabla u|^2 \tag{2}$$

for all u with $Lu = 0$ on B_{2r}.

This is very similar to a result for harmonic functions from Lecture 5, but we will not give the proof here. Instead we will work on some of the consequences. it is clear from (1) that if u is L harmonic (ie $Lu = 0$) on $B_{2^m s}$ then

$$\int_{B_{2^m s}} u^2 \geq (1+c)^m \int_{B_s} u^2. \tag{3}$$

We can rewrite $(1+c)^m = e^{m\log(1+c)} = 2^{m\frac{\log(1+c)}{\log 2}}$. Define $\alpha = \frac{\log(1+c)}{\log 2} > 0$, and let $t = 2^m s$. then

$$\int_{B_t} u^2 \geq \left(\frac{t}{s}\right)^{\alpha} \int_{B_s} u^2. \tag{4}$$

It runs out that a slightly weaker version of (4) holds for all t, not just when t/m is a power of 2.

Proposition 1.2 *Let $t > s > 0$ be real numbers. If u is L harmonic on B_t then*

$$\int_{B_t} u^2 \geq \left(\frac{t}{2s}\right)^{\alpha} \int_{B_s} u^2. \tag{5}$$

Proof Take $2^m s \leq t \leq 2^{m+1}s$. Since u^2 is positive we have

$$\int_{B_{2^m s}} u \leq \int_{B_t} u. \tag{6}$$

We can estimate this bound by (4), so

$$2^{m\alpha} \int_{B_s} u \leq \int_{B_t} u. \tag{7}$$

Also

$$2^{m\alpha} = 2^{(m+1)\alpha} s^{-\alpha} \geq 2^{-\alpha} \left(\frac{t}{s}\right)^{\alpha}.$$

Plugging this into (7) gives

$$\int_{B_t} u^2 \geq \left(\frac{t}{2s}\right)^{\alpha} \int_{B_s} u^2 \tag{8}$$

as required. ∎

We can do exactly the same calculation for the Dirichlet energy to give

Proposition 1.3 *Let $t > s > 0$ be real numbers and let $\beta = \frac{\log(1+d)}{\log 2}$. If u is L harmonic on B_t then*

$$\int_{B_t} |\nabla u^2| \geq \left(\frac{t}{2s}\right)^{\beta} \int_{B_s} |\nabla u|^2. \tag{9}$$

There is a nice corollary to this

Corollary 1.4 *Take $u : \mathbb{R}^n \to \mathbb{R}$ an L harmonic function. If*

$$\int_{\mathbb{R}^n} u^2 < \infty$$

then $u = 0$ on $\mathbb{R}^n$.

Proof Suppose u is not identically zero, say $u(x_0) \neq 0$. Then

$$\int_{B_1(x_0)} u^2 = \epsilon > 0,$$

so

$$\int_{B_t(x_0)} u^2 = \left(\frac{t}{2s}\right)^\alpha \epsilon,$$

which goes to infinity as t gets large.

Now we will prove one of the two inequalities at the start of this lecture. Recall from the proof of the Cacciopolli inequality last time that if u is L harmonic on B_{2r} then

$$4\left(\frac{\Lambda}{\lambda}\right)^2 \int_{B_{2r}} u^2|\nabla\phi|^2 \geq \int_{B_{2r}} \phi^2|\nabla u|^2 \tag{10}$$

for all $\phi \geq 0$ with $\phi = 0$ on the boundary. Also note that

$$|\nabla(\phi u)|^2 = |u\nabla\phi + \phi\nabla u|^2 \leq 2|u\nabla\phi^2| + 2|\phi\nabla u|^2$$

by Cauchy-Schwarz. Putting these together gives

$$\begin{aligned} \int_{B_{2r}} |\nabla(\phi u)|^2 &\leq \int_{B_{2r}} 2|u\nabla\phi^2| + 2|\phi\nabla u|^2 & (11)\\ &\leq 2\left(4\left(\frac{\Lambda}{\lambda}\right)^2 + 1\right)\int_{B_{2r}} u^2|\nabla\phi|^2. & (12) \end{aligned}$$

Notice that ϕu is zero on the boundary of B_{2r}, so the Dirichlet-Poincare applies, and we have

$$\frac{c(n)}{r^2}\int_{B_{2r}} (\phi u)^2 \leq \int_{B_{2r}} |\nabla(\phi u)|^2, \tag{13}$$

so

$$\frac{c(n)}{r^2}\int_{B_{2r}} (\phi u)^2 \leq 2\left(4\left(\frac{\Lambda}{\lambda}\right)^2 + 1\right)\int_{B_{2r}} u^2|\nabla\phi|^2. \tag{14}$$

Pick

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leq r; \\ \frac{2r-|x|}{r} & \text{if } r < x \leq 2r \end{cases}$$

as usual to give

$$\frac{c(n)}{r^2}\int_{B_r} u^2 \leq \frac{2}{r^2}\left(4\left(\frac{\Lambda}{\lambda}\right)^2 + 1\right)\int_{B_{2r}\setminus B_r} u^2. \tag{15}$$

Therefore

$$\left(1+\frac{c(n)}{2\left(\left(2\frac{\Lambda}{\lambda}\right)^2+1\right)}\right)\int_{B_r}u^2\leq\int_{B_{2r}}u^2 \tag{16}$$

as expected.