Homework 6; due Thursday, Nov. 21

1. Let A be a block 2 by 2 matrix over a supercommutative ring R with A_{11}, A_{22} being square matrices of sizes $n \geq 0$, $m \geq 0$ with even entries, and A_{12}, A_{21} having odd entries. The supertrace of A is $\mathrm{str}A := \mathrm{tr}A_{11} - \mathrm{tr}A_{22}$

(a) Show that $\mathrm{str}(AB) = \mathrm{str}(BA)$ for A, B as above. Is this satisfied for the usual trace?
(b) Show that $e^{\mathrm{str}A} = \mathrm{Ber}(e^A)$.

2. Let Y be the real superspace of matrices as in problem 1, which are symmetric in the supersense (i.e. A_{11} is symmetric, A_{22} skew, and $A_{12}^T = A_{21}$), and $Y_+ \subset Y$ be the superdomain of those matrices for which $A_{11} > 0$. Let dA be a supervolume element on Y. Let f be a compactly supported smooth function on Y_+. Show that,

$$\int_{Y_+ \times \mathbb{R}^{n|m}} f(A) e^{-x^T A_{11} x - 2x^t A_{12} \xi - \xi^T A_{22} \xi} dA dx (d\xi)^{-1} = C \int_{Y_+} f(A) \mathrm{Ber}(A)^{-1/2} dA.$$

(C is a constant). What is C?

3. Prove the Amitsur-Levitzki identity: if $X_1, \ldots, X_{2n}$ are n by n matrices over a commutative ring, then $\sum_{\sigma \in S_{2n}} \mathrm{sign}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(2n)} = 0$.

Hint.

(a) Show that for any n by n matrix X with anticommuting entries, $X^{2n} = 0$. (show that traces of X^{2k} vanish for all positive k, then use Hamilton-Cayley).
(b) Apply it to $X = \sum X_i \xi_i$, where ξ_i are anticommuting variables.