

11. FREE FIELD THEORIES IN HIGHER DIMENSIONS

11.1. Minkowski and Euclidean space. Now we pass from quantum mechanics to quantum field theory in dimensions $d \geq 1$. As we explained above, we have two main settings.

1. Minkowski space. Fields are functions on a spacetime V_M , which is a real inner product space of signature $(1, d-1)$. This is where physical processes actually “take place”. The symmetry group of V , $G = SO(1, d-1)$, is called the *Lorentz group*; it is the group of transformation of spacetime in special relativity. Therefore, field theories in Minkowski space which are in an appropriate sense “compatible” with the action of G are called *relativistic*.

Recall some standard facts and definitions. *The light cone* in V is the cone described by the equation $|\mathbf{v}|^2 = 0$, where $|\mathbf{v}|^2 := (\mathbf{v}, \mathbf{v})$. Vectors belonging to the light cone are called lightlike. The light cone divides the space V into spacelike vectors $|\mathbf{v}|^2 < 0$ (the outside of the cone), and timelike vectors $|\mathbf{v}|^2 > 0$ (inside the cone). We will choose one of the two components of the interior of the cone and call it positive; it will be denoted by V_+ . The opposite (negative) component is denoted by V_- . The group of $g \in SO(V) = SO(1, d-1)$ which preserve V_+ is denoted by $SO_+(1, d-1)$; it is the connected component of the identity of the group $SO(1, d-1)$ (which has two connected components).

Often (e.g. when doing Hamiltonian field theory) it is necessary to split V in an orthogonal direct sum $V = V_s \oplus \mathbb{R}$ of space and time. In this decomposition, the space V_s is required to be spacelike (i.e. negative definite), which implies that the time axis \mathbb{R} has to be timelike (positive definite). Note that such a splitting is not unique, and that fixing it breaks the Lorentz symmetry $SO_+(1, d-1)$ down to the usual rotation group $SO(d-1)$. To do explicit calculations, one further chooses Cartesian coordinates x_1, \dots, x_{d-1} on V_s and t on the time axis \mathbb{R} .

2. Euclidean space. Fields are functions on a spacetime V_E , which is a positive definite inner product space. It plays an auxiliary role and has no direct physical meaning, although path integrals computed in this space are similar to expectation values in statistical mechanics.

The two settings are related by the “Wick rotation”. Namely the Euclidean space V_E corresponding to the Minkowski space V_M is the real subspace in $(V_M)\mathbb{C}$ consisting of vectors $(it, x_1, \dots, x_{d-1})$, where t and x_j are real. In other words, to pass to the Euclidean space one needs to make a change of variable $t \rightarrow it$. Note that under this change, the standard metric on the Minkowski space, $dt^2 - \sum dx_i^2$, goes into a negative definite metric $-dt^2 - \sum dx_i^2$. However, the minus sign is traditionally dropped and one considers instead the positive metric $dt^2 + \sum dx_i^2$ on V_E .

11.2. Free scalar bosons. Consider the theory of a free scalar bosonic field ϕ of mass m . The procedure of quantization of this theory in the Lagrangian setting is a straightforward generalization from the case of quantum mechanics. Namely, the Lagrangian for this theory in Minkowski space is $\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2)$, and the Euler-Lagrange equation is $\square\phi = -m^2\phi$, where \square is the D’Alambertian, $\square = \frac{\partial^2}{\partial t^2} - \sum_i \frac{\partial^2}{\partial x_i^2}$. Thus to define the corresponding quantum theory, we should invert the operator $\square + m^2$. As in the quantum mechanics case, this operator is not invertible (0 is in the spectrum), so we should proceed using the Wick rotation.

After the Wick rotation (i.e. transformation $t \rightarrow it$), we arrive at the following Euclidean Lagrangian: $\mathcal{L}_E = \frac{1}{2}((d\phi)^2 + m^2\phi^2)$, and the Euler-Lagrange equation is $\Delta\phi = m^2\phi$. So to define the quantum theory, i.e. the path integral

$$\int \phi(x_1) \cdots \phi(x_n) e^{-S(\phi)} D\phi$$

(where $S = \int \mathcal{L}$), we now need to invert the operator $A = -\Delta + m^2$. The operator A^{-1} is an integral operator, whose kernel is $G(x-y)$, where $G(x)$ is the Green’s function, i.e. the fundamental solution of the differential equation

$$-\Delta G + m^2 G = \delta$$

To solve this equation, note that the solution is rotationally invariant. Therefore, outside of the origin, $G(x) = g(|x|)$, where g is a function on $(0, \infty)$ such that

$$-g'' - \frac{d-1}{r}g' + m^2g = 0$$

(where the left hand side is the radial part of the operator A). This is a version of the Bessel equation. If $m > 0$, the basis of solutions is $r^{\frac{2-d}{2}} J_{\pm \frac{2-d}{2}}(imr)$. (Actually, these functions are elementary for odd d). There exists a unique up to scaling solution, which decays at infinity, namely,

$$g = Cr^{\frac{2-d}{2}} (J_{\frac{2-d}{2}}(imr) + i^d J_{-\frac{2-d}{2}}(imr)), (d \neq 2).$$

For $d = 2$, this expression is zero, and one should instead take the limit of the right hand side divided by $d - 2$ as $d \rightarrow 2$ (which will generate a logarithmic factor $\ln r$). The normalizing constant C can be found from the condition that $AG = \delta$.

Remark. It is easy to check that for $d = 1$ this function equals the familiar Green's function for quantum mechanics, $e^{-mr}/2m$.

If $m = 0$ (massless case), the basis of solutions is: $1, r$ for $d = 1$, $1, \ln r$ for $d = 2$, and $1, r^{2-d}$ for $d > 2$. Thus, if $d \leq 2$, we don't have a decaying solution and thus the corresponding quantum theory will be deficient: it will not satisfy the clustering property. On the other hand, for $d > 2$ we have a unique up to scaling decaying solution $g = Cr^{1-d}$. The constant C is found as in the massive case.

The higher correlation functions are found from the 2-point function via the Wick formula, as usual.

We should now note a fundamental difference between quantum mechanics and quantum field theory in $d > 1$ dimensions. This difference comes from the fact that while for $d = 1$, the Green's function $G(x)$ is continuous at $x = 0$, for $d > 1$ it is singular at $x = 0$. Namely, $G(x)$ behaves like $C|x|^{2-d}$ as $x \rightarrow 0$ for $d > 2$, and as $C \ln|x|$ as $d = 2$. Thus for $d > 1$, unlike the case $d = 1$, the path integral

$$\int \phi(x_1) \cdots \phi(x_n) e^{-S(\phi)} D\phi$$

(as defined above) makes sense only if $x_i \neq x_j$. In other words, this path integral should be regarded not as a function but rather as a distribution. Luckily, there is a canonical way to do it, since the Green's function $G(x)$ is locally L^1 .

Now we can Wick rotate this theory back into the Minkowski space. It is clear that the Green's function will then turn into $G_M(x) = g(\sqrt{|x|^2 + i\varepsilon})$, The higher correlation functions, as before, are determined from this by the Wick formula.

Actually, it is more convenient to describe this theory "in momentum space", where the Green's function can be written more explicitly. Namely, the Fourier transform $\hat{G}(p)$ of the distribution $G(x)$ is a solution of the equation

$$p^2 \hat{G} + m^2 \hat{G} = 1,$$

obtained by Fourier transforming the differential equation for G . Thus,

$$\hat{G}(p) = \frac{1}{p^2 + m^2},$$

as in the quantum mechanics case. Therefore, like in quantum mechanics, the Wick rotation produces the distribution

$$\hat{G}_M(p) = \frac{i}{p^2 - m^2 + i\varepsilon},$$

which is the Fourier transform of $G_M(x)$.

11.3. Spinors. To consider field theory for fermions, we must generalize to the case of $d > 1$ the basic fermionic Lagrangian $\frac{1}{2}\psi \frac{d\psi}{dt}$. To do this, we must replace $\frac{d}{dt}$ by some differential operator on V . This operator should be of first order, since in fermionic quantum mechanics it was important that the equations of motion are first order equations. Clearly, it is impossible to define such an operator so that the Lagrangian is $SO_+(V)$ -invariant, if ψ is a scalar valued (odd) function on V . Thus, a fermionic field in field theory of dimension $d > 1$ cannot be scalar valued, but rather must take values in a real representation S of $SO_+(V)$, such that there exists a nonzero intertwining operator $V \rightarrow \text{Sym}^2 S^*$. This property is satisfied by spinor representations. They are indeed basic in fermionic field theory, and we will now briefly discuss them (for more detail see "Spinors" by P.Deligne, in "QFT and string theory: a course for mathematicians").

First consider the complex case. Let V be a complex inner product space of dimension $d > 1$. Let $Cl(V)$ be the Clifford algebra of V , defined by the relation $\xi\eta + \eta\xi = 2(\xi, \eta)$, $\xi, \eta \in V$. As we discussed,

for even d it is simple and has a unique irreducible representation S of dimension $2^{d/2}$, while for odd d it has two such representations S', S'' of dimension $2^{(d-1)/2}$. It is easy to show that the space $CL_2(V)$ of quadratic elements of $CL(V)$ (i.e. the subspace spanned elements of the form $\xi\eta - \eta\xi, \xi, \eta \in V$) is closed under bracket, and constitutes the Lie algebra $\mathfrak{o}(V)$. Thus $\mathfrak{o}(V)$ acts on S (respectively, S', S''). This action does not integrate to an action of $SO(V)$, but integrates to an action of its double cover $\text{Spin}(V)$.

If d is even, the representation S of $\text{Spin}(V)$ is not irreducible. Namely, recall that S is the exterior algebra of a Lagrangian subspace of V . Thus it splits in a direct sum $S = S_+ \oplus S_-$ (odd and even elements). The subspaces S_+, S_- are subrepresentations of S , which are irreducible. They are called half-spin representations.

If d is odd, the representations S' and S'' of $\text{Spin}(V)$ are irreducible and isomorphic. Any of them will be denoted by S and called the spinor representation. Thus, we have the spinor representation S for both odd and even d , but for even d it is reducible.

An important structure attached to the spinor representation S is the intertwining operator $\Gamma : V \rightarrow \text{End}S$, given by the action of $V \subset Cl(V)$ in S . This intertwiner allows us to define the Dirac operator

$$\mathbf{D} = \sum \Gamma_i \frac{\partial}{\partial x_i}$$

where x_i are coordinates on V associated to an orthonormal basis e_i , and $\Gamma_i = \Gamma(e_i)$. This operator acts on functions from V to S , and $\mathbf{D}^2 = \Delta$, so \mathbf{D} is a square root of the Laplacian. The matrices Γ_i are called Γ matrices.

Note that for even d , one has $\Gamma(v) : S_{\pm} \rightarrow S_{\mp}$, so \mathbf{D} acts from functions with values in S_{\pm} to functions with values in S_{\mp} .

By a spinor representation of $\text{Spin}(V)$ we will mean any linear combination of S_+, S_- for even d , and any multiple of S for odd d . Thus, for every spinor representation we have $\Gamma(v) : Y \rightarrow Y'$, where $S'_+ := S_-$ and $S'_- := S_+$, and $S' := S$.

Now assume that V is a real inner product space with Minkowski metric. In this case we can define the group $\text{Spin}_+(V)$ to be the preimage of $SO_+(V)$ under the map $\text{Spin}(V_{\mathbb{C}}) \rightarrow SO(V_{\mathbb{C}})$. It is a double cover of $SO_+(V)$ (if $d = 2$, this double cover is disconnected and actually a direct product by $\mathbb{Z}/2$).

By a real spinor representation of $SO_+(V)$ we will mean a real representation Y such that $Y_{\mathbb{C}}$ is a spinor representation of $\text{Spin}(V_{\mathbb{C}})$.

11.4. Fermionic Lagrangians. Now let us consider Lagrangians for a spinor field ψ with values in a spinor representation Y . As the Lagrangian is supposed to be real in the Minkowski setting, we will require in that case that Y is real. First of all, let us see what we need in order to write the “kinetic term” $(\psi, \mathbf{D}\psi)$. Clearly, to define such a term (so that it is nonzero), we need an invariant nondegenerate pairing $(,)$ between Y and Y' such that for any $v \in V$, the bilinear form $(x, \Gamma(v)y)$ on Y is symmetric.

Let us find for which Y this is possible (for complex V). The behavior of Spin groups depends on d modulo 8. Thus we will list the answers labeling them by $d \bmod 8$ (they are easily extracted from the tables given in Deligne’s text).

- 0. $n(S_+ \oplus S_-)$.
- 1. nS .
- 2. $nS_+ \oplus kS_-$.
- 3. nS .
- 4. $n(S_+ \oplus S_-)$.
- 5. $2nS$.
- 6. $2nS_+ \oplus 2kS_-$.
- 7. $2S$.

Let us now find when we can also add a mass term. Recall that the mass term has the form $(\psi, M\psi)$, so it corresponds to an invariant skew-symmetric operator $M : Y \rightarrow Y^*$. Let us list those Y from the above list, for which such nondegenerate operator exists.

- 0. $2n(S_+ \oplus S_-)$.
- 1. $2nS$.

2. $n(S_+ \oplus S_-)$.
3. nS .
4. $n(S_+ \oplus S_-)$.
5. $2nS$.
6. $2n(S_+ \oplus S_-)$.
7. $2S$.

To pass to the real Minkowski space (in both massless and massive case), one should put the additional requirement that Y should be a real representation.

We note that upon Wick rotation to Minkowski space, it may turn out that a real spinor representation Y will turn into a complex representation which has no real structure. Namely, this happens for massless spinors that take values in S_{\pm} if $d = 2 \pmod 8$. These representations have a real structure for Minkowskian V (i.e. for $SO_+(1, d-1)$), but no real structure for Euclidean V (i.e. for $SO(d)$). This is quite obvious, for example, when $d = 2$ (check!).

One may think that this causes a problem in quantum field theory, where we would be puzzled what to integrate over – real or complex space. However, the problem in fact does not arise, since we have to integrate over fermions, and integration over fermions (say, in the finite dimensional case) is purely algebraic and does not make a distinction between real and complex.

11.5. Free fermions. Let us now consider a free theory for a spinor $\psi : V \rightarrow \Pi Y$, where Y is a spinor representation, defined by a Lagrangian $\mathcal{L} = \frac{1}{2}((\psi, \mathbf{D}\psi) - (\psi, M\psi))$, where M is allowed to be degenerate (we assume that Y is such that this expression makes sense). The equation of motion in Minkowski space is $\mathbf{D}\psi = M\psi$, where we have identified Y' and Y^* using the pairing $Y \times Y' \rightarrow \mathbb{R}$ used to define the kinetic term. Thus, to define the corresponding quantum theory, we need to invert the operator $\mathbf{D} - M$. As usual, this cannot be done because of a singularity, and we have to consider the Wick rotation.

The Wick rotation produces the Euclidean Lagrangian $\mathcal{L} = \frac{1}{2}((\psi, \mathbf{D}_E\psi) + (\psi, M\psi))$ (note that the i in the kinetic term is hidden in the definition of the Euclidean Dirac operator). We invert $\mathbf{D}_E + M$, to obtain the Euclidean Green's function. To do this, it is convenient to go to momentum space, i.e. perform a Fourier transform. Namely, after Fourier transform \mathbf{D}_E turns into the operator $i\mathbf{p}$, where $\mathbf{p} = \sum p_j \Gamma_j$, and p_j are the operators of multiplication by the coordinates p_j . Thus, the Green's function (i.e. the 2-point function) $G(x) \in \text{Hom}(Y', Y)$ is the Fourier transform of the matrix-valued function $\frac{1}{i\mathbf{p} + M}$. Let $M^\dagger : Y' \rightarrow Y$ be the operator such that $\Gamma_i M = M^\dagger \Gamma_i$. Then, $(-i\mathbf{p} + M^\dagger)(i\mathbf{p} + M) = p^2 + M^\dagger M$, so that

$$\hat{G}(p) = (p^2 + M^\dagger M)^{-1}(-i\mathbf{p} + M^\dagger).$$

This shows that $G(x)$ is expressed through the Green's function in the bosonic case by differentiations (how?). After Wick rotation back to the Minkowski space, we get

$$\hat{G}_M(p) = (p^2 - M^\dagger M + i\varepsilon)^{-1}(\mathbf{p} + iM^\dagger).$$

Finally, the higher correlation functions, as usual, are found from the Wick formula.