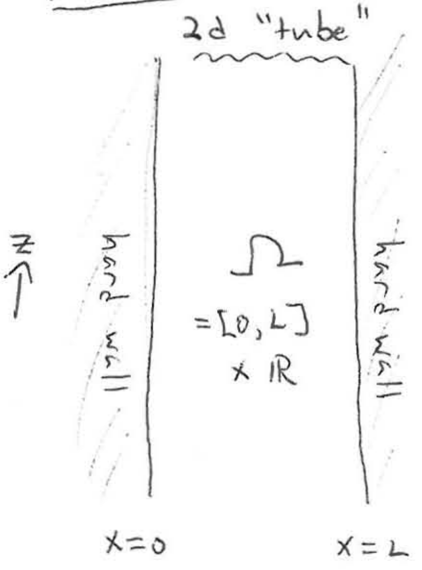
Waveguide Modes and Evanescent Waves



$$\hat{A} = \nabla^2$$

$$u\big|_{\partial\Omega} = 0$$

$$\Downarrow$$

$$\hat{A} = \hat{A}^* < 0$$

under $\langle u, v\rangle = \int_\Omega \bar{u} v$

waves: $\hat{A} u = \dfrac{\partial^2 u}{\partial t^2}$

normal modes: $\hat{A} u = \lambda u = -\omega^2 u$

translational symmetry in z

$\Rightarrow$ separable eigensolutions

$$u(x, z) = e^{ikz} u_k(x) \text{ :}$$

$$\hat{A}\, e^{ikz} u_k = -\omega^2(k)\, e^{ikz} u_k$$

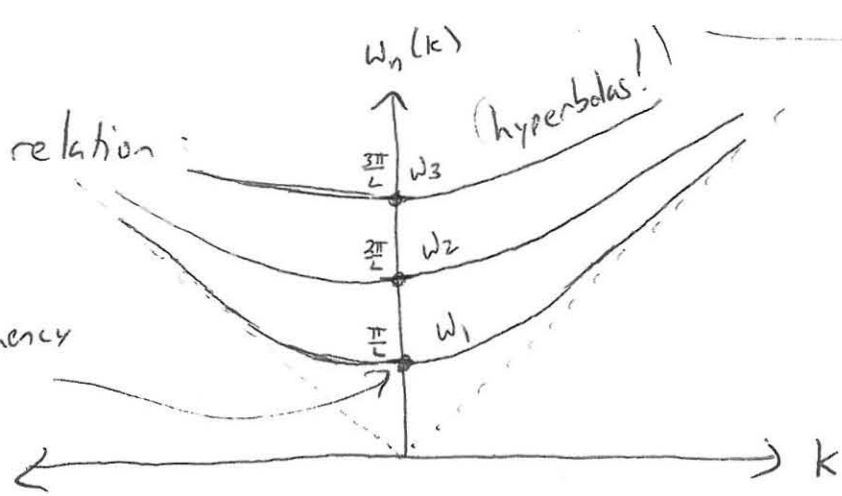
$$\Rightarrow \underbrace{\left(e^{-ikz} \hat{A}\, e^{ikz}\right)}_{\hat{A}_k} u_k = -\omega^2 u_k$$

Solution: $e^{-ikz}\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}\right)\left(e^{ikz} u_k(x)\right)$

$$= \cancel{e^{-ikz}}\, \cancel{e^{ikz}} \left(\frac{\partial^2}{\partial x^2} - k^2\right) u_k \quad = -\omega^2 u_k$$

$$\Rightarrow \quad u_k'' = -(\omega^2 - k^2)\, u_k \quad \Rightarrow \quad \boxed{\begin{aligned} u_{k,n}(x) &= \sin\left(\frac{n\pi x}{L}\right) \\ \omega_n(k) &= \pm\sqrt{\left(\frac{n\pi}{L}\right)^2 + k^2} \end{aligned}}$$

Dispersion relation:



each mode has a low-frequency cutoff

(also called "β" = "propagation constant")

as $k \to 0$ (cutoff):

$V_g = \dfrac{d\omega}{dk} \to 0$ (standing waves)

$V_p = \dfrac{\omega}{k} \to \infty$

* Evanescent waves: exponentially decaying / growing solutions
for real $\omega \Rightarrow k$:

$$k_n = \pm\sqrt{\omega^2 - \left(\frac{n\pi}{L}\right)^2} = \begin{cases} \text{real } (\underline{\text{propagating}}) & |\omega| \geq \frac{n\pi}{L}: \text{above cutoff} \\ \text{imaginary } (\underline{\text{evanescent}}) & |\omega| < \frac{n\pi}{L}: \text{below cutoff} \end{cases}$$

— <u>not allowed</u> in ∞ waveguide <u>unless</u> we break translational symmetry

... we never admit solutions that grow exponentially as $z \to \pm\infty$

— <u>needed</u> if we break translation invariance by changing geometry <u>or</u> by introducing a <u>source</u>

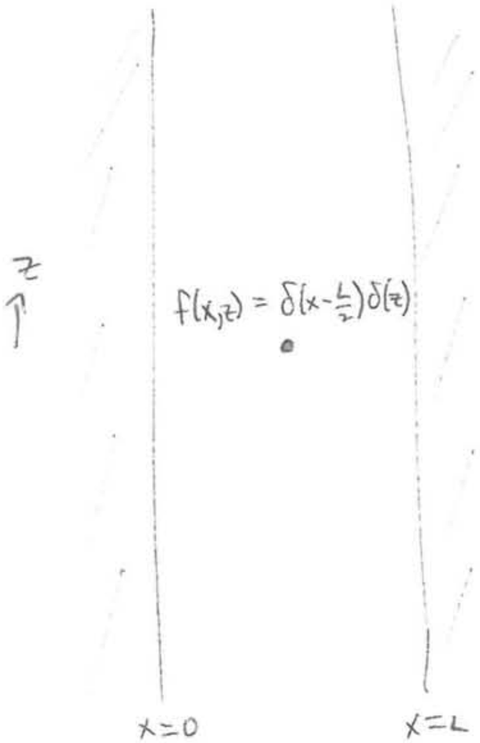
* Scalar <u>Helmholtz equation</u>:

$$\hat{A} u = \frac{\partial^2 u}{\partial t^2} + \underbrace{f(x,t)}_{\text{source term}} \xrightarrow[\text{in } t]{\text{Fourier transform}} \hat{A}\hat{u} = -\omega^2 \hat{u} + \hat{f}$$

(equivalently: time-harmonic response $\hat{u}e^{-i\omega t}$ to a time-harmonic source $\hat{f}e^{-i\omega t}$)

$$\Rightarrow \boxed{\left(\hat{A} + \omega^2\right) u(\vec{x}) = f(\vec{x})} \quad \text{[dropping the hats]}$$

example: δ- function source f in $(\hat{A}+w^2)u = f$



look for solutions: (sum of propagating + evanescent)

$$\boxed{u(x,z) = \begin{cases} \sum_n c_n^+ \sin\left(\frac{n\pi x}{L}\right) e^{+ik_n z} & z>0 \\ \sum_n c_n^- \sin\left(\frac{n\pi x}{L}\right) e^{-ik_n z} & z<0 \end{cases} \qquad \text{where } k_n = \sqrt{w^2 - \left(\frac{n\pi}{L}\right)^2}}$$

$\Rightarrow$ by construction,

$$\boxed{(\hat{A}+w^2)\,u = 0 \quad \text{for} \quad z \neq 0}$$

$$\left[= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + w^2\right) \sum_n c_n^\pm \sin\left(\frac{n\pi x}{L}\right) e^{\pm ik_n z} \right.$$

$$\left. = \sum_n c_n^\pm \sin\left(\frac{n\pi x}{L}\right) e^{\pm ik_n z} \underbrace{\left(-\left(\frac{n\pi}{L}\right)^2 - k_n^2 + w^2\right)}_{0} \right]$$

* u is continuous at $z=0$, if $\boxed{c_n^+ = c_n^-} = c_n$

but $\frac{\partial u}{\partial z}$ is discontinuous

$$\Rightarrow \frac{\partial^2 u}{\partial z^2} = \delta(z) \sum_n c_n \sin\left(\frac{n\pi x}{L}\right) \cdot 2ik_n \qquad \left(\text{only source of a } \delta(z)!\right)$$

$$/\!/$$

$$= \delta\left(x - \frac{L}{2}\right)\delta(z) \quad (= f(x))$$

$$\Rightarrow c_n = \frac{\cancel{2}}{L} \frac{1}{\cancel{2}ik_n} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \delta\left(x-\frac{L}{2}\right) dx = \frac{1}{ik_n L} \sin\left(\frac{n\pi \cancel{L}}{2\cancel{L}}\right)$$

$$= \frac{1}{ik_n L} \begin{cases} 0 & n \text{ even (antisymmetric } \sin\left(\frac{n\pi x}{L}\right)) \\ (-1)^{\frac{n-1}{2}} & n \text{ odd (symmetric } \sin) \end{cases}$$

$$\Longrightarrow \boxed{u(x,z) = \sum_{\substack{n \\ \text{odd}}} \frac{(-1)^{\frac{n-1}{2}}}{iL\underbrace{\sqrt{w^2-\left(\frac{n\pi}{L}\right)^2}}_{=k_n}} \sin\left(\frac{n\pi x}{L}\right) e^{ik_n|z|}}$$

* sum of <u>waves</u> <u>propagating</u> <u>away</u> from source and <u>evanescent</u> waves <u>decaying</u> <u>away</u> from source

* solution <u>blows</u> <u>up</u> as $\omega \to$ cutoff $\frac{n\pi}{L}$ of any mode!
(real effect!)

— <u>but</u> remember that this is generally just a single Fourier component $\hat{u}$ of solution:

suppose $f(x,t) = \delta(z)\,\delta(x-\frac{L}{2})\,f(t)$

for some "pulse" $f(t)$
(usually square-integrable)

$\xleftrightarrow{\text{Fourier}} \hat{f}(\omega)$

$$\Longrightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega) e^{-i\omega t}\, d\omega$$

where $$\hat{u} = \sum_{\text{odd } n} \frac{(-1)^{\frac{n+1}{2}}}{ik_n L} \sin\left(\frac{n\pi x}{L}\right) e^{ik_n|z|}\, \hat{f}(\omega)$$

and this <u>integral</u> is <u>finite</u> because $\frac{1}{k_n}$ is an <u>integrable</u> <u>singularity</u> ($\int \frac{1}{\sqrt{\omega}}\, d\omega$ is finite)