Problems for The 1-D Heat Equation

18.303 Linear Partial Differential Equations

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Fall 2006

1. A bar with initial temperature profile $f(x) > 0$, with ends held at 0^o C, will cool as $t \to \infty$, and approach a steady-state temperature 0^oC. However, whether or not all parts of the bar start cooling initially depends on the shape of the initial temperature profile. The following example may enable you to discover the relationship.

   (a) Find an initial temperature profile $f(x)$, $0 \le x \le 1$, which is a linear combination of $\sin \pi x$ and $\sin 3\pi x$, and satisfies $\frac{df}{dx}(0) = 0 = \frac{df}{dx}(1)$, $f\left(\frac{1}{2}\right) = 2$.

   (b) Solve the problem

   $$u_t = u_{xx}; \qquad u(0,t) = 0 = u(1,t); \qquad u(x,0) = f(x).$$

   Note: you can just write down the solution we had in class, but make sure you know how to get it!

   (c) Show that for some x, $0 \le x \le 1$, $u_t(x,0)$ is positive and for others it is negative. How is the sign of $u_t(x,0)$ related to the shape of the initial temperature profile? How is the sign of $u_t(x,t)$, $t > 0$, related to subsequent temperature profiles? Graph the temperature profile for $t = 0, 0.2, 0.5, 1$ on the same axis (you may use Matlab).

2. Initial temperature pulse. Solve the inhomogeneous heat problem with mixed boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad u(0,t) = 0 = u(1,t); \qquad u(x,0) = P_w(x)$$

where $t > 0$, $0 \le x \le 1$, and

$$P_w(x) = \begin{cases} 0 & \text{if } 0 < x < \frac{1}{2} - \frac{w}{2} \\ \frac{u_0}{w} & \text{if } \frac{1}{2} - \frac{w}{2} < x < \frac{1}{2} + \frac{w}{2} \\ 0 & \text{if } \frac{1}{2} + \frac{w}{2} < x < 1 \end{cases} \tag{1}$$

Note: we derived the form of the solution in class. You may simply use this and replace $P_w(x)$ with $f(x)$.

(a) Show that the temperature at the midpoint of the rod when $t = 1/\pi^2$ (dimensionless) is approximated by

$$u\left(\frac{1}{2}, \frac{1}{\pi^2}\right) \approx \frac{2u_0}{e}\left(\frac{\sin(\pi w/2)}{\pi w/2}\right)$$

Can you distinguish between a pulse with width $w = 1/1000$ from one with $w = 1/2000$, say, by measuring $u\left(\frac{1}{2}, \frac{1}{\pi^2}\right)$?

(b) Illustrate the solution qualitatively by sketching (i) some typical temperature profiles in the $u - t$ plane (i.e. $x =$ constant) and in the $u - x$ plane (i.e. $t =$ constant), and (ii) some typical level curves $u(x,t) =$ constant in the $x - t$ plane. At what points of the set $D = \{(x,t) : 0 \le x \le 1, t \ge 0\}$ is $u(x,t)$ discontinuous?

3. Consider the homogeneous heat problem with type II BCs:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(1,t); \qquad u(x,0) = f(x)$$

where $t > 0$, $0 \le x \le 1$ and f is a piecewise smooth function on $[0,1]$.

(a) Find the eigenvalues λ_n and the eigenfunctions $X_n(x)$ for this problem. Write the formal solution of the problem (a), and express the constant coefficients as integrals involving $f(x)$.

(b) Find a series solution in the case that $f(x) = u_0$, u_0 a constant. Find an approximate solution good for large times. Sketch temperature profiles (u vs. x) for different times.

(c) Evaluate $\lim_{t\to\infty} u(x,t)$ for the solution (a) when $f(x) = P_w(x)$ with $P_w(x)$ defined in (1). Illustrate the solution qualitatively by sketching temperature profiles and level curves as in Problem 2(b). It is not necessary to find the complete formal solution.

4. Consider the homogeneous heat problem with type II BCs:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad \frac{\partial u}{\partial x}(0,t) = 0 = u(1,t); \qquad u(x,0) = f(x)$$

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