Solutions to Problems for The 1-D Heat Equation

18.303 Linear Partial Differential Equations

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1. A bar with initial temperature profile $f(x) > 0$, with ends held at 0^o C, will cool as $t \to \infty$, and approach a steady-state temperature 0^oC. However, whether or not all parts of the bar start cooling initially depends on the shape of the initial temperature profile. The following example may enable you to discover the relationship.

   (a) Find an initial temperature profile $f(x)$, $0 \le x \le 1$, which is a linear combination of $\sin \pi x$ and $\sin 3\pi x$, and satisfies $\frac{df}{dx}(0) = 0 = \frac{df}{dx}(1)$, $f\left(\frac{1}{2}\right) = 2$.

   Solution: A linear combination of $\sin \pi x$ and $\sin 3\pi x$ is

$$f(x) = a \sin 3\pi x + b \sin \pi x$$

   Imposing the conditions gives

$$\begin{aligned} 0 &= \frac{df}{dx}(0) = \pi(3a + b) \\ 0 &= \frac{df}{dx}(1) = -\pi(3a + b) \\ 2 &= f\left(\frac{1}{2}\right) = -a + b \end{aligned}$$

   The first two equations yield the same thing, $3a = -b$. Substituting this into the last equation gives

$$a = -\frac{1}{2}, \qquad b = \frac{3}{2}.$$

   Thus

$$f(x) = -\frac{1}{2} \sin 3\pi x + \frac{3}{2} \sin \pi x \tag{1}$$

(b) Solve the problem

$$u_t = u_{xx}; \qquad u(0,t) = 0 = u(1,t); \qquad u(x,0) = f(x).$$

Note: you can just write down the solution we had in class, but make sure you know how to get it!

Solution: This is the basic heat problem we considered in class, with solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2\pi^2 t} \tag{2}$$

where

$$B_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx \tag{3}$$

and $f(x)$ is given in (1). The form of (1) is already a sine series, with $B_1 = 3/2$, $B_3 = -1/2$ and $B_n = 0$ for all other n. You can check this for yourself by computing integrals in (3) for $f(x)$ given by (1), from the orthogonality of $\sin n\pi x$. Therefore,

$$u(x,t) = \frac{3}{2}\sin(\pi x) e^{-\pi^2 t} - \frac{1}{2}\sin(3\pi x) e^{-9\pi^2 t} \tag{4}$$

(c) Show that for some x, $0 \le x \le 1$, $u_t(x,0)$ is positive and for others it is negative. How is the sign of $u_t(x,0)$ related to the shape of the initial temperature profile? How is the sign of $u_t(x,t)$, $t > 0$, related to subsequent temperature profiles? Graph the temperature profile for $t = 0, 0.2, 0.5, 1$ on the same axis (you may use Matlab).

Solution: Differentiating $u(x,t)$ in time gives

$$u_t(x,t) = -\pi^2\left(\frac{3}{2}\sin(\pi x) e^{-\pi^2 t} - \frac{9}{2}\sin(3\pi x) e^{-9\pi^2 t}\right)$$

Setting $t = 0$ gives

$$u_t(x,0) = -\frac{3}{2}\pi^2(\sin(\pi x) - 3\sin(3\pi x))$$

Note that

$$u_t\left(\frac{1}{6},0\right) = \frac{15}{4}\pi^2 > 0, \qquad u_t\left(\frac{1}{2},0\right) = -6\pi^2 < 0$$

Thus at $x = 1/6$, u_t is positive and for $x = 1/2$, u_t is negative.

From the PDE,

$$u_t = u_{xx}$$

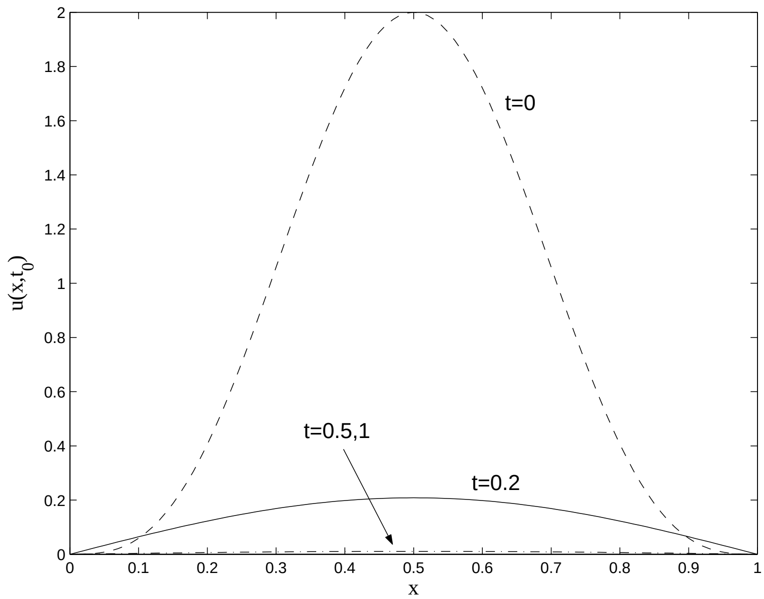

Figure 1: Plots of $u(x, t_0)$ for $t_0 = 0, 0.2, 0.5, 1$.

and hence the sign of u_t gives the concavity of the temperature profile $u(x, t_0)$, t_0 constant. Note that for $u_{xx}(x, t_0) > 0$, the profile $u(x, t_0)$ is concave up, and for $u_{xx}(x, t_0) < 0$, the profile $u(x, t_0)$ is concave down. At $t_0 = 0$, the sign of $u_t(x, 0)$ give the concavity of the initial temperature profile $u(x, 0) = f(x)$.

In Figure 1, $u(x, t_0)$ is plotted for $t_0 = 0, 0.2, 0.5, 1$.

2. Initial temperature pulse. Solve the inhomogeneous heat problem with Type I boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad u(0, t) = 0 = u(1, t); \qquad u(x, 0) = P_w(x)$$

where $t > 0$, $0 \leq x \leq 1$, and

$$P_w(x) = \begin{cases} 0 & \text{if } 0 < x < \frac{1}{2} - \frac{w}{2} \\ \frac{u_0}{w} & \text{if } \frac{1}{2} - \frac{w}{2} < x < \frac{1}{2} + \frac{w}{2} \\ 0 & \text{if } \frac{1}{2} + \frac{w}{2} < x < 1 \end{cases} \tag{5}$$

Note: we derived the form of the solution in class. You may simply use this and replace $P_w(x)$ with $f(x)$.

(a) Show that the temperature at the midpoint of the rod when $t = 1/\pi^2$ (dimensionless) is approximated by

$$u\left(\frac{1}{2}, \frac{1}{\pi^2}\right) \approx \frac{2u_0}{e}\left(\frac{\sin(\pi w/2)}{\pi w/2}\right)$$

Can you distinguish between a pulse with width $w = 1/1000$ from one with $w = 1/2000$, say, by measuring $u\left(\frac{1}{2}, \frac{1}{\pi^2}\right)$?

(b) Illustrate the solution qualitatively by sketching (i) some typical temperature profiles in the $u - t$ plane (i.e. $x =$ constant) and in the $u - x$ plane (i.e. $t =$ constant), and (ii) some typical level curves $u(x,t) =$ constant in the $x - t$ plane. At what points of the set $D = \{(x,t) : 0 \leq x \leq 1, t \geq 0\}$ is $u(x,t)$ discontinuous?

Solution: This is the Heat Problem with Type I homogeneous BCs. The solution we derived in class is, with $f(x)$ replaced by $P_w(x)$,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t} \tag{6}$$

where the B_n's are the Fourier coefficients of $f(x) = P_w(x)$, given by

$$B_n = 2\int_0^1 P_w(x) \sin(n\pi x)\, dx$$

Breaking the integral into three pieces and substituting for $P_\varepsilon(x)$ from (5) gives

$$\begin{aligned}
B_n &= 2\int_0^{1/2-w/2} P_w(x)\sin(n\pi x)\,dx + 2\int_{1/2-w/2}^{1/2+w/2} P_w(x)\sin(n\pi x)\,dx \\
&\quad +2\int_{1/2+w/2}^{1} P_w(x)\sin(n\pi x)\,dx \\
&= 0 + 2\int_{1/2-w/2}^{1/2+w/2} \frac{u_0}{w}\sin(n\pi x)\,dx + 0 \\
&= \frac{2u_0}{w}\left\{-\frac{\cos(n\pi x)}{n\pi}\right\}_{1/2-w/2}^{1/2+w/2} \\
&= u_0\frac{\cos\left(\frac{n\pi}{2}(1-w)\right) - \cos\left(\frac{n\pi}{2}(1+w)\right)}{wn\pi/2} \qquad (7)
\end{aligned}$$

We apply the cosine rule

$$\cos(r-s) - \cos(r+s) = 2\sin r \sin s$$

with $r = n\pi/2$, $s = n\pi w/2$ to Eq. (7),

$$B_n = \frac{4u_0}{wn\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi w}{2}$$

When n is even (and nonzero), i.e. $n = 2m$ for some integer m,

$$B_{2m} = \frac{2u_0}{wm\pi}\sin m\pi \sin m\pi w = 0$$

When n is odd, i.e. $n = 2m - 1$ for some integer m,

$$B_{2m-1} = 2u_0 (-1)^{m+1} \frac{\sin((2m-1)\pi w/2)}{(2m-1)\pi w/2}. \tag{8}$$

(a) The temperature at the midpoint of the rod, $x = 1/2$, at scaled time $t = 1/\pi^2$ is, from (6) and (8),

$$\begin{aligned} u(x,t) &= \sum_{m=1}^{\infty} 2u_0 (-1)^{m+1} \frac{\sin((2m-1)\pi w/2)}{(2m-1)\pi w/2} \sin\left((2m-1)\frac{\pi}{2}\right) e^{-(2m-1)^2} \\ &= \sum_{m=1}^{\infty} \frac{2u_0}{e^{(2m-1)^2}} \left(\frac{\sin((2m-1)\pi w/2)}{(2m-1)\pi w/2}\right). \end{aligned}$$

For $t \geq 1/\pi^2$, the first term gives a good approximation to $u(x,t)$,

$$u\left(\frac{1}{2}, \frac{1}{\pi^2}\right) \approx u_1\left(\frac{1}{2}, \frac{1}{\pi^2}\right) = \frac{2u_0}{e}\left(\frac{\sin(\pi w/2)}{\pi w/2}\right).$$

To distinguish between pulses with $\varepsilon = 1/1000$ and $w = 1/2000$, note that $\lim_{w\to 0} \frac{\sin \pi w/2}{\pi w/2} = 1$, and so for smaller and smaller w, the corresponding temperature $u\left(\frac{1}{2}, \frac{1}{\pi^2}\right)$ gets closer and closer to $2u_0/e$,

$$u\left(\frac{1}{2}, \frac{1}{\pi^2}\right) \approx u_1\left(\frac{1}{2}, \frac{1}{\pi^2}\right) = \frac{2u_0}{e}\left(1 - \frac{\pi^2\varepsilon^2}{2\cdot 3!} + \cdots\right), \qquad w \ll 1.$$

In particular,

$$\begin{aligned} & u_1\left(\frac{1}{2}, \frac{1}{\pi^2}; w = \frac{1}{1000}\right) - u_1\left(\frac{1}{2}, \frac{1}{\pi^2}; w = \frac{1}{2000}\right) \\ = \; & \frac{2u_0}{e}\left(\frac{\sin(\pi/2000)}{\pi/2000} - \frac{\sin(\pi/4000)}{\pi/4000}\right) \\ \approx \; & -\frac{2u_0}{e} \times 3.1 \times 10^{-7} \end{aligned}$$

Thus it is hard to distinguish these two temperature distributions, at least by measuring the temperature at the center of the rod at time $t = 1/\pi^2$. By this time, diffusion has smoothed out some of the details of the initial condition.

(b) The solution $u(x,t)$ is discontinuous at $t = 0$ at the points $x = (1 \pm w)/2$. That said, $u(x,t)$ is piecewise continuous on the entire interval $[0,1]$. Thus, the Fourier series for $u(x,0)$ converges everywhere on the interval and equals $u(x,0)$ at all points except $x = (1 \pm w)/2$. The temperature profiles ($u-t$ plane, $u-x$ plane), 3D solution and level curves are shown.

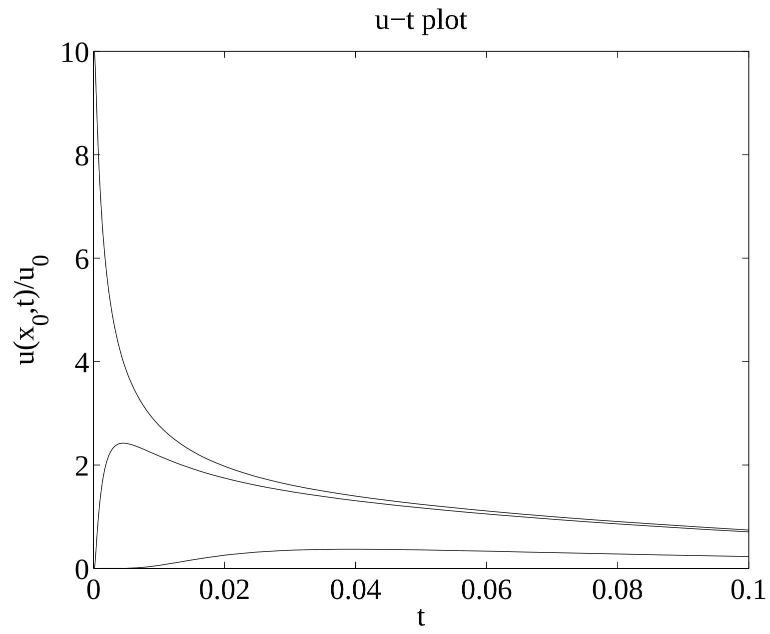

Figure 2: Time temperature profiles $u(x_0, t)$ at $x_0 = 0.5$, 0.4 and 0.1 (from top to bottom). The t-axis is the time profile corresponding to $x_0 = 0$, 1.

Recall that to draw the level curves, it is easiest to already have drawn the spatial temperature profiles. Draw a few horizontal broken lines across your u vs. x plot. Suppose you draw a horizontal line $u = u_1$. Suppose this line $u = u_1$ crosses one of your profiles $u(x, t_0)$ at position $x = x_1$. Then (x_1, t_0) is a point on the level curve $u(x, t) = u_1$. Now plot this point in your level curve plot. By observing where the line $u = u_1$ crosses your various spatial profiles, you fill in the level curve $u(x, t) = u_1$. Repeat this process for a few values of u_1 to obtain a few representative level curves. Plot also the special case level curves: $u(x, t) = u_0/w$, $u(x, t) = 0$, etc. Come and see me if you still have problems.

3. Consider the homogeneous heat problem with type II BCs:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad \frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(1, t); \qquad u(x, 0) = f(x) \tag{9}$$

where $t > 0$, $0 \le x \le 1$ and f is a piecewise smooth function on $[0, 1]$.

(a) Find the eigenvalues λ_n and the eigenfunctions $X_n(x)$ for this problem. Write the formal solution of the problem (a), and express the constant coefficients as integrals involving $f(x)$.

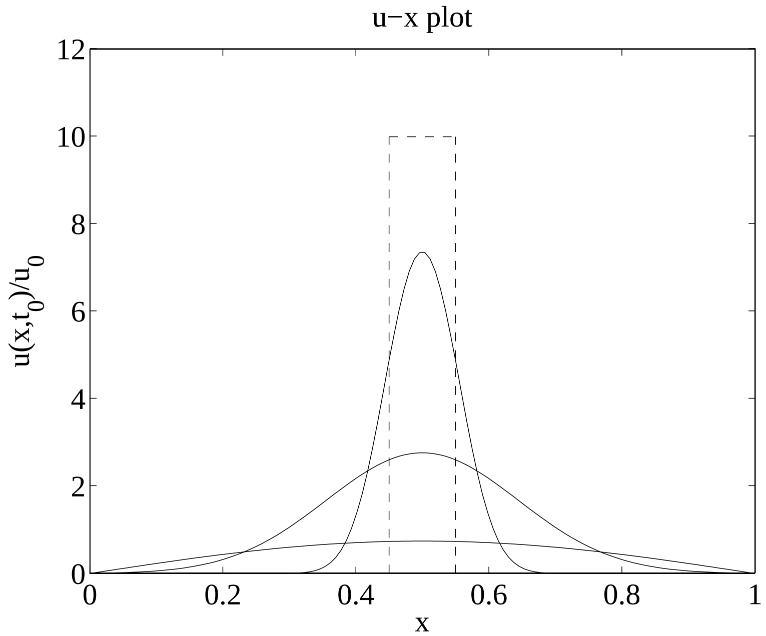

Figure 3: Spatial temperature profiles $u(x, t_0)$ at $t_0 = 0$ (dash), 0.001, 0.01, 0.1. The x-axis from 0 to 1 is the limiting temperature profile $u(x, t_0)$ as $t_0 \to \infty$.

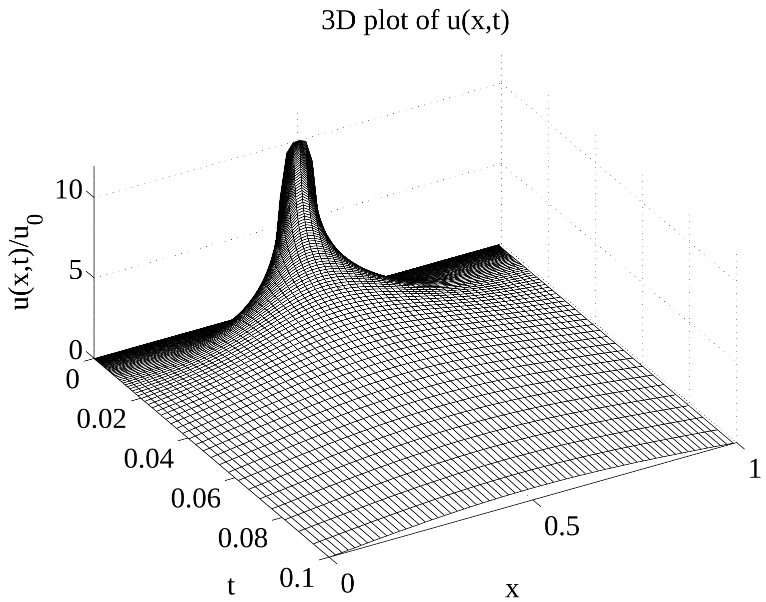

Figure 4:

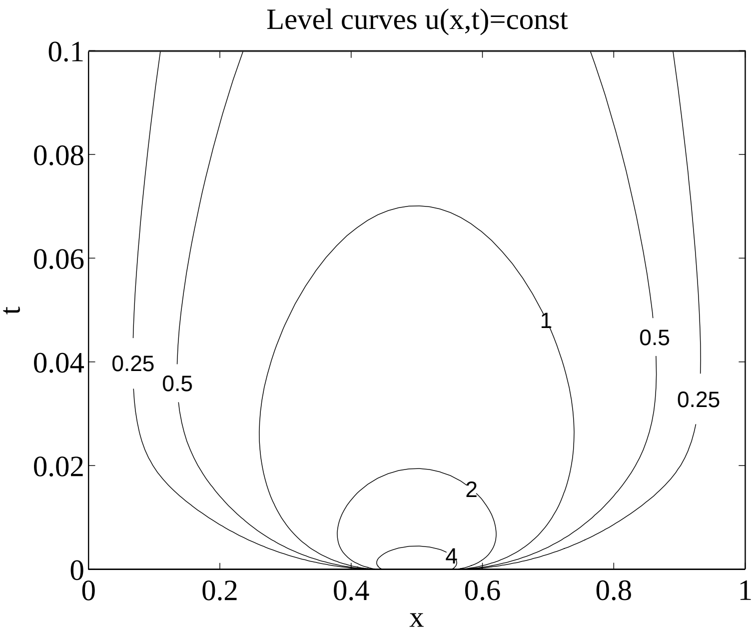

Figure 5: Level curves $u(x,t)/u_0 = C$ for various values of the constant C. Numbers adjacent to curves indicate the value of C. The line segment $(1-w)/2 \leq x \leq (1+w)/2$ at $t=0$ is the level curve with $C = 1/w = 10$. The lines $x=0$ and $x=1$ are also level curves with $C=0$.

(b) Find a series solution in the case that $f(x) = u_0$, u_0 a constant. Find an approximate solution good for large times. Sketch temperature profiles (u vs. x) for different times.

(c) Evaluate $\lim_{t\to\infty} u(x,t)$ for the solution (a) when $f(x) = P_w(x)$ with $P_w(x)$ defined in (5). Illustrate the solution qualitatively by sketching temperature profiles and level curves as in Problem 2(b). It is not necessary to find the complete formal solution.

Solution: (a) To find a series solution, we use separation of variables,

$$u(x,t) = X(x)\,T(t) \tag{10}$$

The PDE in (9) gives the usual

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

where λ is constant since the left hand side is a function of x only and the middle is a function of t only. Substituting (10) into the BCs in (9) gives

$$X'(0) = X'(1) = 0$$

The Sturm-Liouville boundary value problem for $X(x)$ is thus

$$X'' + \lambda X = 0; \qquad X'(0) = X'(1) = 0 \tag{11}$$

Let us try $\lambda < 0$. Then the solutions are

$$X(x) = c_1 e^{-\sqrt{|\lambda|}x} + c_2 e^{\sqrt{|\lambda|}x}$$

and imposing the BCs gives

$$\begin{aligned} 0 &= X'(0) = -\sqrt{|\lambda|}c_1 + \sqrt{|\lambda|}c_2 \\ 0 &= X'(1) = -\sqrt{|\lambda|}c_1 e^{-\sqrt{|\lambda|}} + \sqrt{|\lambda|}c_2 e^{\sqrt{|\lambda|}} \end{aligned}$$

The first equation gives $c_1 = -c_2$ and substituting this into the second, we have

$$0 = \sqrt{|\lambda|}c_2 \left(e^{-\sqrt{|\lambda|}} + e^{\sqrt{|\lambda|}}\right)$$

Since $\lambda < 0$, the bracketed expression is positive. Hence $c_1 = c_2 = 0$, i.e. $X(x)$ must be the trivial solution, and we discard the case $\lambda < 0$.

For $\lambda = 0$, $X(x) = c_1 x + c_2$ and both BCs are satisfied by taking $c_1 = 0$. Thus $X(x) = c_2 = A_0$ (we'll use A_0 by convention - it's just another way to name the constant). Hence, the case $\lambda = 0$ is allowed and yields a non-trivial solution.

For $\lambda > 0$, we have

$$X = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x$$

The BC $X'(0) = 0$ implies $c_1 = 0$. The other BC implies

$$0 = X'(1) = -c_2\sqrt{\lambda}\sin\sqrt{\lambda}$$

For a non-trivial solution, c_2 must be nonzero. Since $\lambda > 0$ then we must have $\sin\sqrt{\lambda} = 0$, which implies the eigenvalues are

$$\lambda_n = n^2\pi^2, \qquad n = 1, 2, 3, ...$$

and the eigenfunctions are

$$X_n(x) = \cos(n\pi x)$$

For each n, the solution for $T(t)$ is $T_n(t) = e^{-\lambda_n t}$. Hence the series solution for $u(x,t)$ is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \exp\left(-n^2\pi^2 t\right) \tag{12}$$

At $t = 0$, the initial condition gives

$$f(x) = u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \tag{13}$$

The orthogonality conditions are found using the identity

$$2\cos(n\pi x)\cos(m\pi x) = \cos((m-n)\pi x) + \cos((m+n)\pi x)$$

Note also that for $m, n = 1, 2, 3...$, we have

$$\int_0^1 \cos((m-n)\pi x)\,dx = \begin{cases} 1 & m = n \neq 0 \\ 0 & m \neq n \end{cases}$$

$$\int_0^1 \cos((m+n)\pi x)\,dx = 0$$

The last integral follows since $m+n$ cannot be zero for any positive integers m, n. Combining the three previous equations gives the orthogonality conditions

$$\int_0^1 \cos(n\pi x)\cos(m\pi x)\,dx = \begin{cases} 1/2 & m = n \neq 0 \\ 0 & m \neq n \end{cases} \tag{14}$$

Multiplying each side of (13) by $\cos(m\pi x)$, integrating from $x = 0$ to 1, and applying the orthogonality condition (14) gives

$$A_0 = \int_0^1 f(x)\,dx \tag{15}$$

$$A_m = 2\int_0^1 \cos(m\pi x) f(x)\,dx \tag{16}$$

(b) Substituting $f(x) = u_0$ into (16) and (15) gives

$$A_n = 2u_0 \int_0^1 \cos(n\pi x)\,dx = 0, \quad n > 0 \tag{17}$$

$$A_0 = \int_0^1 u_0 dx = u_0$$

It is no surprise that $A_n = 0$ for $n > 0$ since the IC $f(x) = u_0$ is one of the eigenfunctions, $X_0(x) = A_0$. The series solution is simply

$$u(x, t) = u_0 = f(x).$$

Temperature profiles are simply a plot of $f(x) = u_0$, i.e. the temperature along the rod does not change. This is reasonable, since the rod is initially

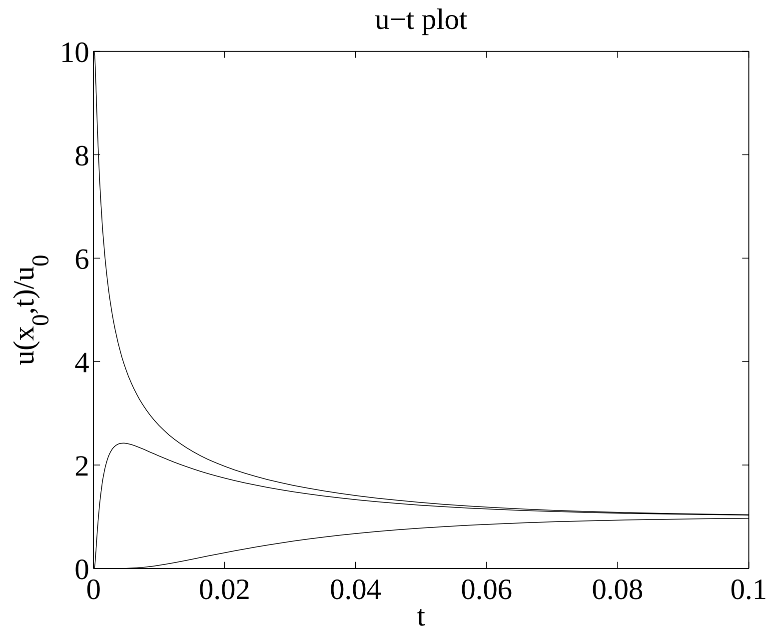

Figure 6: Time temperature profiles $u(x_0, t)$ at $x_0 = 0.5$, 0.4 and 0.1 (from top to bottom).

at a constant temperature and is completely insulated - so nothing will happen.

(c) Taking the limit $t \to \infty$ of (12) and using (15) gives

$$\lim_{t\to\infty} u(x,t) = A_0 = \int_0^1 f(x)\, dx = \int_0^1 P_w(x)\, dx = \int_{1/2-w/2}^{1/2+w/2} \frac{u_0}{w} dx = u_0$$

Thus the temperature along rod eventually becomes the constant u_0. Lastly, we illustrate the solution qualitatively by sketching temperature profiles and level curves in Figures 6 to 9.

4. Consider the homogeneous heat problem with type III (mixed) BCs:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad \frac{\partial u}{\partial x}(0,t) = 0 = u(1,t); \qquad u(x,0) = f(x)$$

where $t > 0$, $0 \le x \le 1$ and f is a piecewise smooth function on $[0, 1]$.

(a) Find the eigenvalues λ_n and the eigenfunctions $X_n(x)$ for this problem. Write the formal solution of the problem (a), and express the constant coefficients as integrals involving $f(x)$.

(b) Find a series solution in the case that $f(x) = u_0$, u_0 a constant. Find an approximate solution good for large times. Sketch temperature profiles (u vs. x) for different times.

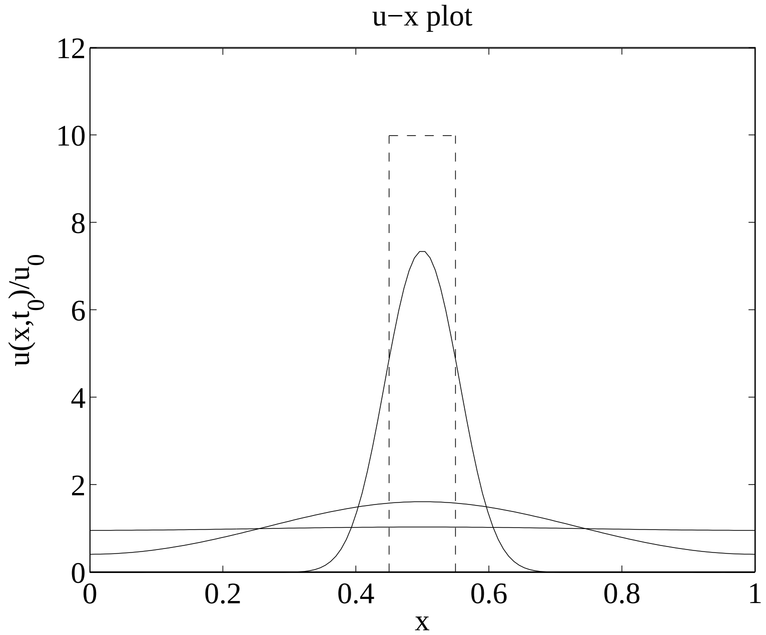

Figure 7: Spatial temperature profiles $u\,(x, t_0)$ at $t_0 = 0$ (dash), 0.001, 0.03, 0.1. The line $u(x, t_0)/u_0 = 1$ is the limiting temperature profile as $t_0 \to \infty$. Note that the ends of the rod heat up!

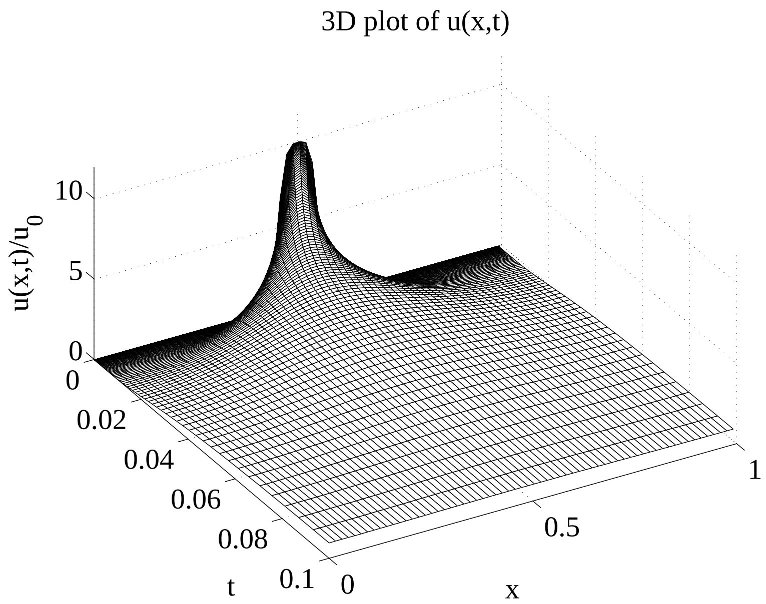

Figure 8:

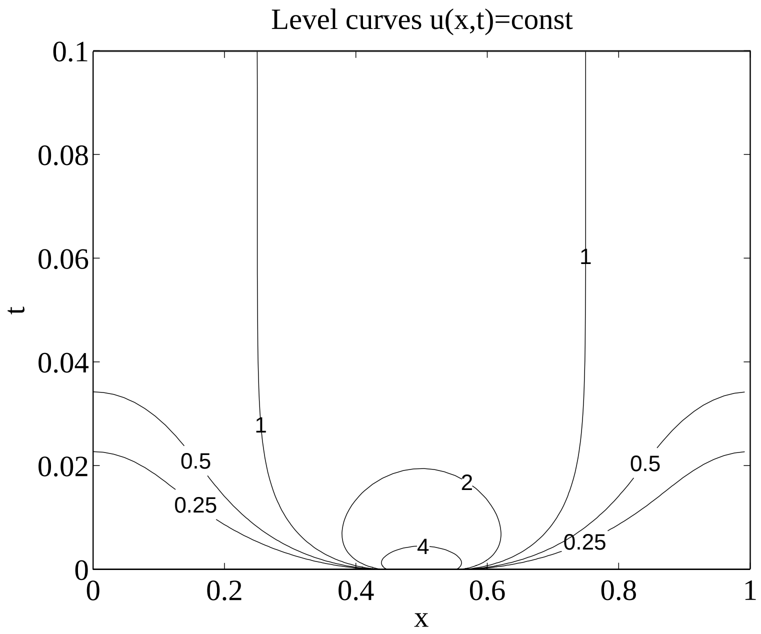

Figure 9: Level curves $u(x,t)/u_0 = C$ for various values of the constant C. Numbers adjacent to curves indicate the value of C. The line segment $(1-w)/2 \le x \le (1+w)/2$ at $t=0$ is the level curve with $C = 1/w = 10$.

(c) Evaluate $\lim_{t\to\infty} u(x,t)$ for the solution (a) when $f(x) = P_w(x)$ with $P_w(x)$ defined in (5). Illustrate the solution qualitatively by sketching temperature profiles and level curves as in Problem 2(b). It is not necessary to find the complete formal solution.

Solution: (a) To find a series solution, we use separation of variables as before (Eq. (10)), but now obtain the Sturm Liouville problem

$$X'' + \lambda X = 0; \qquad X'(0) = X(1) = 0 \tag{18}$$

Let us try $\lambda < 0$. Then the solutions are

$$X(x) = c_1 e^{-\sqrt{|\lambda|}x} + c_2 e^{\sqrt{|\lambda|}x}$$

and imposing the BCs gives

$$\begin{aligned} 0 &= X'(0) = -\sqrt{|\lambda|}c_1 + \sqrt{|\lambda|}c_2 \\ 0 &= X(1) = c_1 e^{-\sqrt{|\lambda|}} + c_2 e^{\sqrt{|\lambda|}} \end{aligned}$$

The first equation gives $c_1 = -c_2$ and substituting this into the second, we have

$$0 = c_2 e^{-\sqrt{|\lambda|}}\left(1 - e^{2\sqrt{|\lambda|}}\right)$$

Since $\lambda < 0$, then $e^{2\sqrt{|\lambda|}} > 1$ and the bracketed expression is negative. Hence $c_2 = c_1 = 0$, i.e. $X(x)$ must be the trivial solution, and we discard the case $\lambda < 0$.

For $\lambda = 0$, $X(x) = c_1 x + c_2$ and imposing the BCs gives $c_1 = c_2 = 0$. Thus we discard the $\lambda = 0$ case.

For $\lambda > 0$, we have

$$X = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x$$

The BC $X'(0) = 0$ implies $c_1 = 0$. The other BC implies

$$0 = X(1) = c_2 \cos \sqrt{\lambda}$$

For a non-trivial solution, c_2 must be nonzero. Since $\lambda > 0$ then we must have $\cos \sqrt{\lambda} = 0$, which implies the eigenvalues are

$$\lambda_n = \frac{(2n-1)^2 \pi^2}{4}, \qquad n = 1, 2, 3, ...$$

and the eigenfunctions are

$$X_n(x) = \cos\left(\frac{2n-1}{2}\pi x\right)$$

For each n, the solution for $T(t)$ is $T_n(t) = e^{-\lambda_n t}$. Hence the series solution for $u(x,t)$ is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n-1}{2}\pi x\right) \exp\left(-\frac{(2n-1)^2 \pi^2}{4} t\right) \tag{19}$$

At $t = 0$,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{(2n-1)\pi}{2}\pi x\right) \tag{20}$$

The orthogonality conditions are found using the identity

$$2\cos\left(\frac{2n-1}{2}\pi x\right)\cos\left(\frac{2m-1}{2}\pi x\right) = \cos((m-n)\pi x) + \cos((m+n-1)\pi x)$$

Note also that for $m, n = 1, 2, 3...$, we have

$$\int_0^1 \cos((m-n)\pi x)\, dx = \begin{cases} 1 & m = n \neq 0 \\ 0 & m \neq n \end{cases}$$

$$\int_0^1 \cos((m+n-1)\pi x)\, dx = 0$$

The last integral follows since $m+n-1$ cannot be zero for any positive integers m, n. Combining the three previous equations gives the orthogonality conditions

$$\int_0^1 \cos\left(\frac{2n-1}{2}\pi x\right)\cos\left(\frac{2m-1}{2}\pi x\right)dx = \begin{cases} 1/2 & m=n\neq 0 \\ 0 & m\neq n \end{cases} \tag{21}$$

Multiplying each side of (20) by $\cos\left(\left(2m-1\right)\pi x/2\right)$, integrating from $x = 0$ to 1, and applying the orthogonality condition (21) gives

$$A_m = 2\int_0^1 \cos\left(\frac{2m-1}{2}\pi x\right) f\left(x\right)dx$$

(b) Substituting $f\left(x\right) = u_0$ into (16) and (15) gives

$$\begin{aligned} A_n &= 2u_0\int_0^1 \cos\left(\frac{2n-1}{2}\pi x\right)dx = 2u_0\left[\frac{\sin\left(\frac{2n-1}{2}\pi x\right)}{\frac{2n-1}{2}\pi}\right]_0^1 \\ &= \frac{4u_0}{\left(2n-1\right)\pi}\sin\left(\frac{2n-1}{2}\pi\right) = \frac{4u_0\left(-1\right)^{n+1}}{\left(2n-1\right)\pi} \end{aligned}$$

Thus the series solution is

$$u\left(x,t\right) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{2n-1}\cos\left(\frac{2n-1}{2}\pi x\right)\exp\left(-\frac{\left(2n-1\right)^2\pi^2}{4}t\right)$$

After $t \geq 4/\pi^2$, we may approximate the series by the first term,

$$u\left(x,t\right) \approx u_1\left(x,t\right) = \frac{4u_0}{\pi}\cos\left(\frac{\pi x}{2}\right)\exp\left(-\frac{\pi^2}{4}t\right)$$

The temperature profiles (u vs. x) for different times are given below in Figures 10 to 13.

(c) Taking the limit $t \to \infty$ of (19) gives

$$\lim_{t\to\infty} u\left(x,t\right) = 0$$

Thus the temperature along rod eventually goes to zero. Lastly, we illustrate the solution qualitatively by sketching temperature profiles and level curves (Figures 14 to 17).

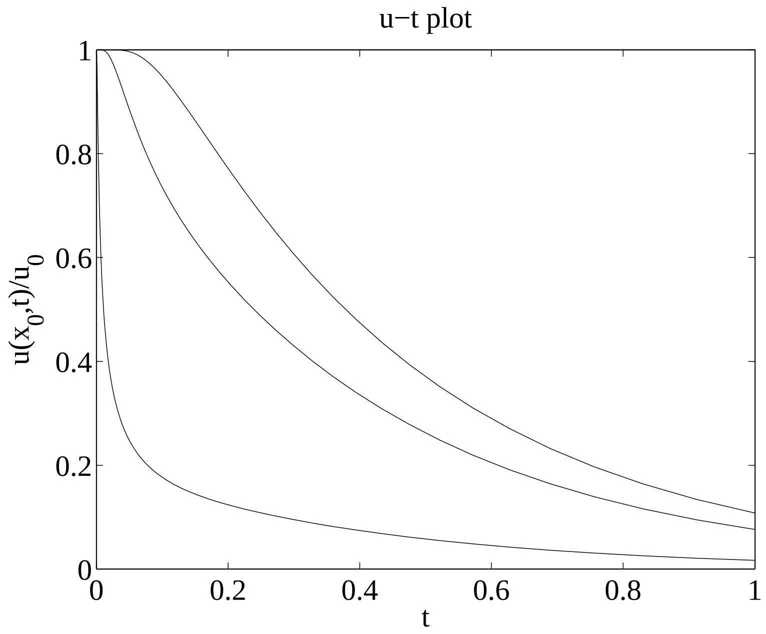

Figure 10: Time temperature profiles $u\,(x_0, t)$ at $x_0 = 0.01$, 0.4 and 0.9 (from top to bottom).

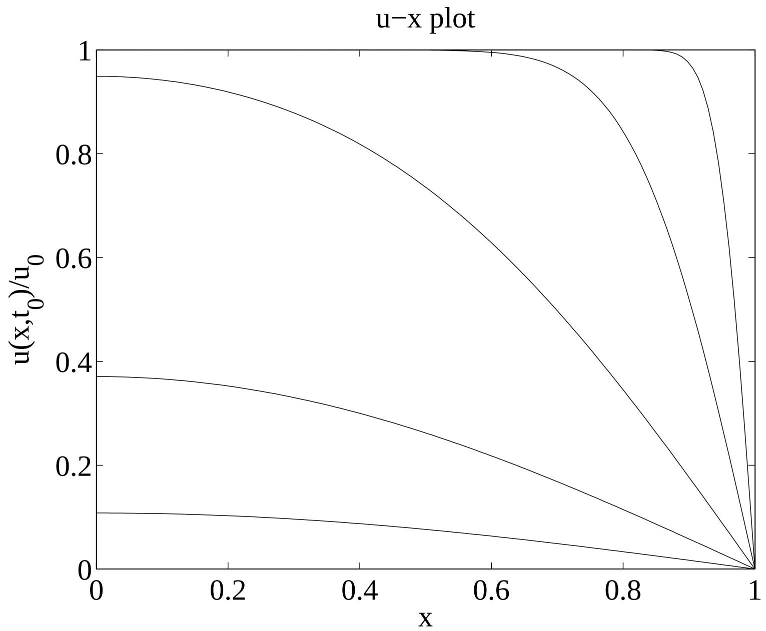

Figure 11: Spatial temperature profiles $u\,(x, t_0)$ at $t = 0.001$, 0.01, 0.1, 0.5 and 1 from top to bottom. The line $u(x, t_0)/u_0 = 1$ is the initial temperature profile at $t = 0$.

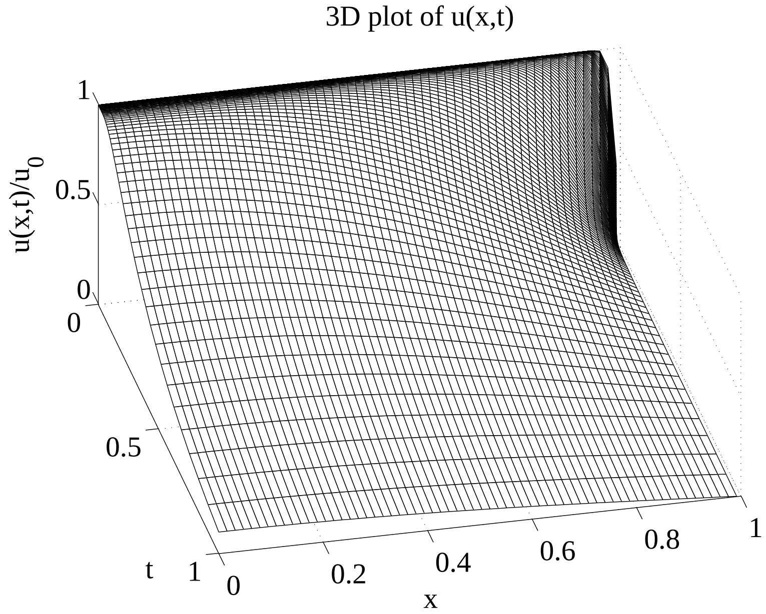

Figure 12:

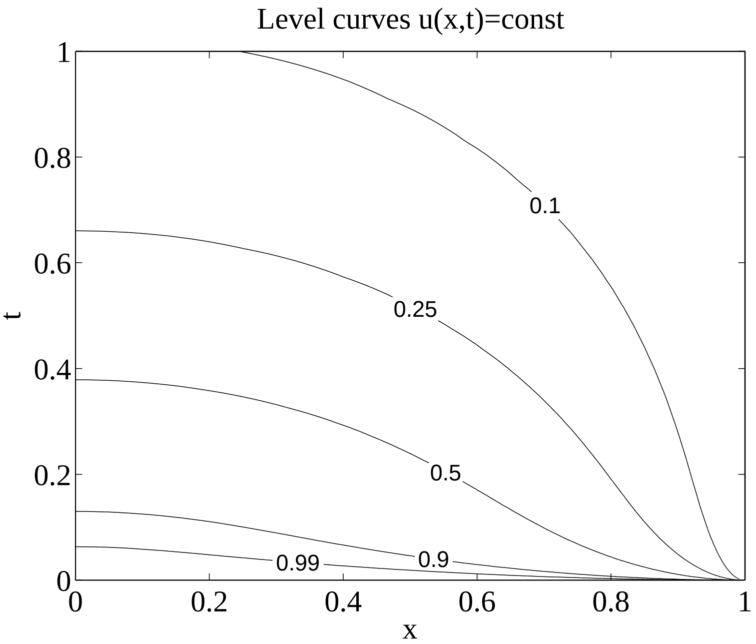

Figure 13: Level curves $u(x,t)/u_0 = C$ for various values of the constant C. Numbers adjacent to curves indicate the value of C.

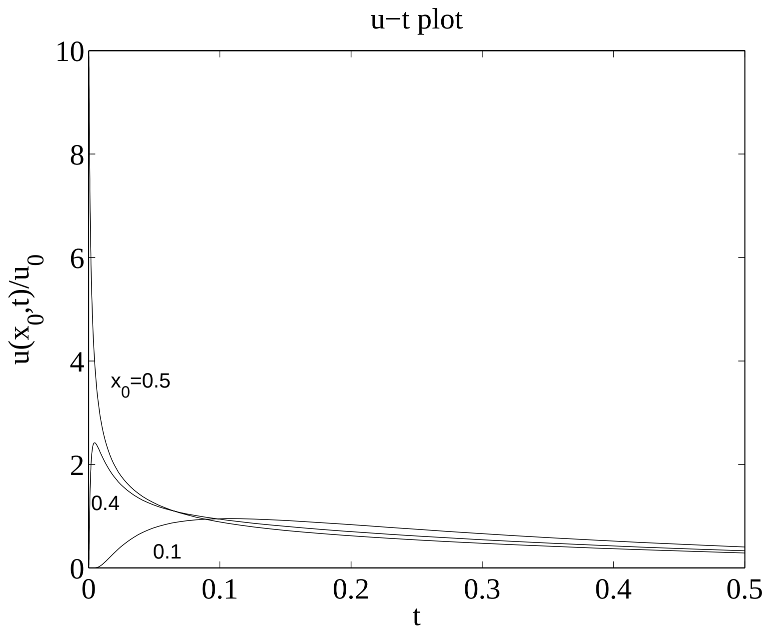

Figure 14: Time temperature profiles $u\left(x_0, t\right)$ at $x_0 = 0.5$, 0.4 and 0.1 (from top to bottom).

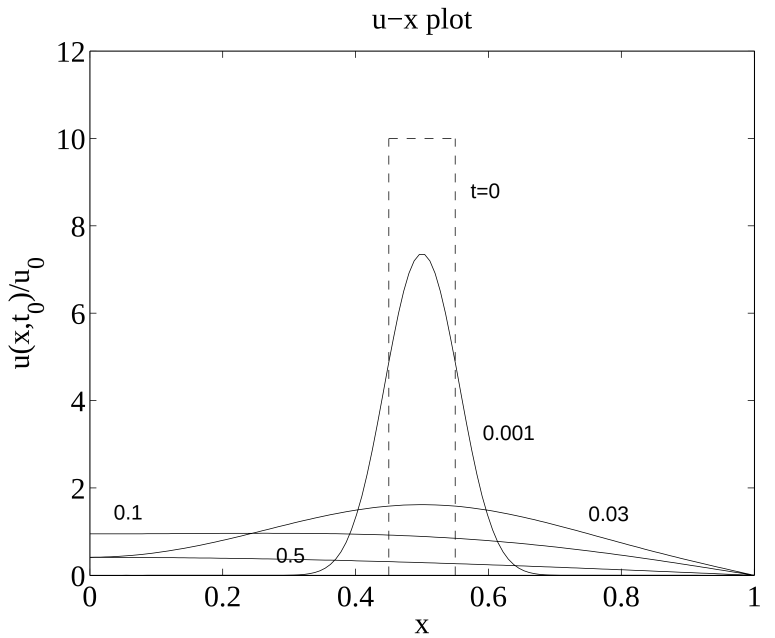

Figure 15: Spatial temperature profiles $u\left(x, t_0\right)$ at $t_0 = 0$ (dash), 0.001, 0.03, 0.1 and 0.5.

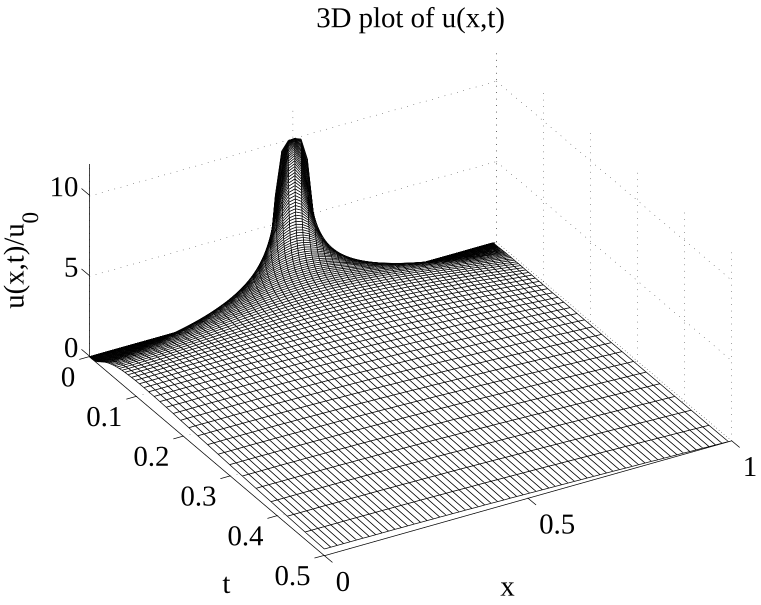

Figure 16:

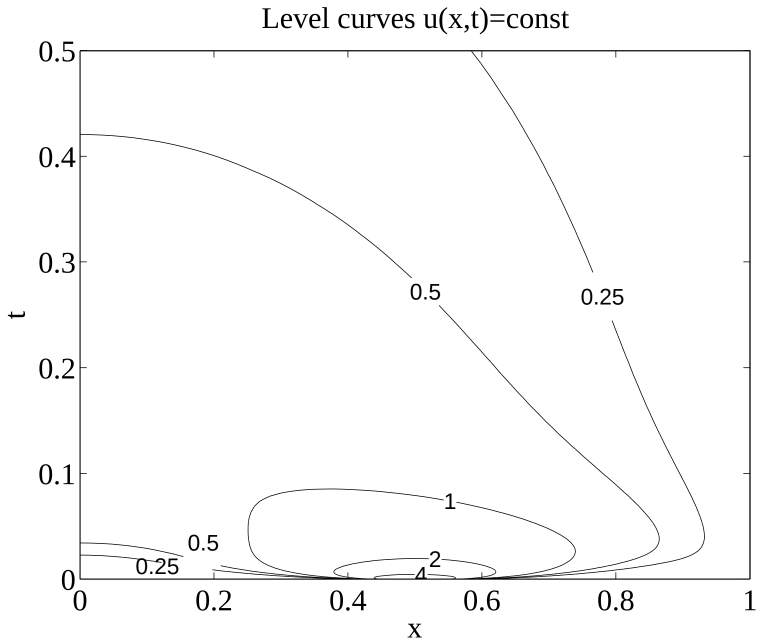

Figure 17: Level curves $u\left(x,t\right)/u_0 = C$ for various values of the constant C. Numbers adjacent to curves indicate the value of C. The line segment $\left(1-w\right)/2 \leq x \leq \left(1+w\right)/2$ at $t=0$ is the level curve with $C = 1/w = 10$.