Solutions to Test 1

18.303 Linear Partial Differential Equations

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1 Rules

You may only use pencils, pens, erasers, and straight edges. No calculators, notes, books or other aides are permitted. Scrap paper will be provided.

Be sure to show a few key intermediate steps when deriving results - answers only will not get full marks.

2 Given

You may assume the eigenvalues of the Sturm-Liouville problem

$$\begin{aligned} X'' + \lambda X &= 0, \qquad 0 < x < 1 \\ X(0) &= 0 \qquad X(1) = 0 \end{aligned}$$

are $\lambda_n = n^2\pi^2$ and $X_n(x) = \sin(n\pi x)$, for $n = 1, 2, ...$, without derivation.

You may also assume the following orthogonality conditions for m, n positive integers:

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

3 Question

Consider the following heat problem in dimensionless variables

$$\begin{aligned} u_t &= u_{xx} - bx, \qquad 0 < x < 1, \qquad t > 0 \\ u(0,t) &= 0, \qquad u(1,t) = 0, \qquad t > 0 \\ u(x,0) &= u_0 \qquad 0 < x < 1, \end{aligned}$$

where $b > 0$ and $u_0 > 0$ are constants. This is the heat equation with a negative source (i.e. extracting heat from the rod).

(a) [3 points] Derive the steady-state (equilibrium) solution

$$u_E(x) = \frac{b}{6} x \left(1 - x^2\right)$$

It is insufficient to simply verify that the solution works.

Solution: The steady-state solution satisfies the PDE and BCs,

$$\begin{aligned} 0 &= u_E'' - bx \\ u_E(0) &= 0 = u_E(1) \end{aligned}$$

Integrating the ODE for u_E gives

$$u_E(x) = \frac{b}{6} x \left(x^2 - 1\right)$$

(b) [3 points] Using $u_E(x)$, transform the given heat problem for $u(x,t)$ into the following problem for a function $v(x,t)$:

$$\begin{aligned} v_t &= v_{xx}, \qquad 0 < x < 1, \qquad t > 0 \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0 \\ v(x,0) &= f(x) \qquad 0 < x < 1. \end{aligned}$$

where $f(x)$ will be determined by the transformation. Write v_t, v_{xx} in terms of u, b, x. State $f(x)$ in terms of u_0, b and x.

Solution: Writing

$$v(x,t) = u(x,t) - u_E(x)$$

we have

$$\begin{aligned} v_t &= u_t \\ v_{xx} &= u_{xx} - u_E'' = u_{xx} - bx \end{aligned}$$

and hence the PDE becomes

$$v_t = v_{xx}$$

The BCs for v are

$$\begin{aligned} v(0,t) &= u(0,t) - u_E(0) = 0 - 0 = 0 \\ v(1,t) &= u(1,t) - u_E(1) = 0 - 0 = 0 \end{aligned}$$

The IC is

$$v(x,0) = u(x,0) - u_E(x) = u_0 - \frac{b}{6}x\left(x^2 - 1\right)$$

(c) [10 points] Derive the solution

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

and derive equations for B_n in terms of $f(x)$. Be sure to give the intermediate steps: separate variables, write down problems and solve for $X(x)$ (using information from the Given section), solve for $T_n(t)$, put things together, impose the IC. Use orthogonality of $\sin(n\pi x)$ (see Given section) to find B_n in terms of $f(x)$. Substitute for $f(x)$ from part (b). You may use (without proof) the fact that

$$\int_0^1 x\left(x^2 - 1\right)\sin(n\pi x)\,dx = \frac{6(-1)^n}{\pi^3 n^3}, \qquad \int_0^1 \sin(n\pi x)\,dx = \frac{1-(-1)^n}{\pi n}$$

Solution: Using separation of variables, we let

$$u(x,t) = X(x)\,T(t)$$

and substitute this into the PDE to obtain

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda$$

where λ is a constant because the left hand side depends only on x and the middle only depends on t.

The Sturm-Liouville problem for $X(x)$ is

$$X'' + \lambda X = 0; \qquad X(0) = 0 = X(1)$$

whose solution is (given),

$$X_n(x) = \sin(n\pi x), \qquad \lambda_n = n^2\pi^2.$$

The equations for $T(t)$ are

$$T_n(t) = B_n e^{-n^2\pi^2 t}$$

and this gives the solution $v_n(x,t)$ to the PDE

$$v_n(x,t) = X_n(x)\, T_n(t) = B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

for constants B_n. Summing all $v_n(x,t)$ together gives

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

Imposing the IC gives

$$v(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

Multiplying by $\sin(m\pi x)$ and integrating from $x = 0$ to $x = 1$ gives

$$\int_0^1 v(x,0) \sin(m\pi x)\, dx = \sum_{n=1}^{\infty} B_n \int_0^1 \sin(n\pi x) \sin(m\pi x)\, dx$$

Using the given orthogonality condition gives

$$B_m = 2\int_0^1 f(x) \sin(m\pi x)\, dx$$

Substituting for $f(x)$ gives

$$\begin{aligned}
B_m &= 2\int_0^1 \left(u_0 - \frac{b}{6} x\left(x^2 - 1\right)\right) \sin(m\pi x)\, dx \\
&= 2u_0 \int_0^1 \sin(m\pi x)\, dx - \frac{b}{3}\int_0^1 x\left(x^2-1\right) \sin(m\pi x)\, dx \\
&= \frac{2u_0\left(1 - (-1)^m\right)}{\pi m} - \frac{2b(-1)^m}{\pi^3 m^3}
\end{aligned}$$

(e) [4 points] Prove that the solution $v(x,t)$ is unique. Recall that $v(x,t)$ satisfies

$$\begin{aligned}
v_t &= v_{xx}, \qquad 0 < x < 1, \qquad t > 0 \\
v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0 \\
v(x,0) &= f(x) \qquad 0 < x < 1.
\end{aligned}$$

Solution: Consider 2 solutions and define $h(x,t) = v_1(x,t) - v_2(x,t)$. Then $h(x,t)$ satisfies

$$\begin{aligned} h_t &= h_{xx}, \qquad 0 < x < 1, \qquad t > 0 \\ h(0,t) &= 0, \qquad h(1,t) = 0, \qquad t > 0 \\ h(x,0) &= 0 \qquad 0 < x < 1. \end{aligned}$$

Define the function

$$H(t) = \int_0^1 h^2(x,t)\,dx$$

Differentiate in time,

$$\frac{dH}{dt} = \int_0^1 2hh_t dx = \int_0^1 2hh_{xx} dx = 2\ hh_x|_0^1 - 2\int_0^1 h_x^2 dx = -2\int_0^1 h_x^2 dx \le 0$$

Also, $H(0) = \int_0^1 0dx = 0$. And $H(t) \ge 0$. Thus $H = 0$ for all time, which implies $h(x,t) = 0$ for all space and time.

Aside: Suppose there was a time t_0 and point x_0 where $h^2(x_0,t_0) > 0$. Then by continuity the integral would not be zero and $H(t) > 0$ there. Thus $h(x,t) = 0$ for all space and time.

(g) [2 points] Solve for

$$u(x,t) = u_E(x) + \sum_{n=1}^{\infty} u_n(x,t) \tag{1}$$

using the earlier transformations.

Solution: Reversing the earlier transformations, we have

$$\begin{aligned} u(x,t) &= u_E(x) + v(x,t) \\ &= \frac{b}{6}x\left(x^2-1\right) + v(x,t) \\ &= \frac{b}{6}x\left(x^2-1\right) + \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t}\sin(n\pi x) \end{aligned}$$

(g) [3 points] Based on the definition of $u_n(x,t)$ in Eq. (1), write down what $u_n(x,t)$ is from your solution to (g). Then assume $u_0 = b/\pi^2$ and show that

$$\left|\frac{u_2(x,t)}{u_1(x,t)}\right| \le \frac{1}{12}e^{-3}, \qquad t \ge 1/\pi^2.$$

Solution: We have

$$u_n(x,t) = B_n e^{-n^2\pi^2 t}\sin(n\pi x)$$

and

$$B_1 = \frac{4u_0}{\pi} + \frac{2b}{\pi^3} = \frac{6b}{\pi^3}, \qquad B_2 = -\frac{2b}{8\pi^3}$$

so that

$$\begin{aligned} \left|\frac{u_2(x,t)}{u_1(x,t)}\right| &= \left|\frac{B_2 e^{-4\pi^2 t}\sin(2\pi x)}{B_1 e^{-\pi^2 t}\sin(\pi x)}\right| = \left|\frac{\frac{2b}{8\pi^3}e^{-3\pi^2 t}2\sin(\pi x)\cos(\pi x)}{\frac{6b}{\pi^3}\sin(\pi x)}\right| \\ &= \left|\frac{1}{12}(\cos\pi x)\, e^{-3\pi^2 t}\right| \le \frac{1}{12}e^{-3\pi^2 t} \le \frac{1}{12}e^{-3} \end{aligned}$$

for $t \ge 1/\pi^2$.

(h) [3 points] In (g) you showed that the second term was small compared to the first, so (without proof) write down the first term approximation

$$u(x,t) \approx u_E(x) + A_1 e^{-\pi^2 t}\sin(\pi x)$$

which is expected to be good for $t \ge 1/\pi^2$. Sketch $u = u_0$ and $u = u_E(x)$ for $0 < x < 1$ and comment on the physical significance of the sign of A_1. You may assume $u_0 = b/\pi^2$.

Solution: The first term approximation is

$$u(x,t) \approx u_E(x) + u_1(x,t) = u_E(x) + B_1 e^{-\pi^2 t}\sin(\pi x) = u_E(x) + \frac{6b}{\pi^3}e^{-\pi^2 t}\sin(\pi x)$$

Thus $A_1 = 6b/\pi^2 > 0$, which means the rod cools down to $u_E(x)$. A plot of $u_0 = b/\pi^2$ and $u_E(x)$ is given below, for $b = 1$.