

Test 2

18.303 Linear Partial Differential Equations

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1 Rules

You may only use pencils, pens, erasers, and straight edges. No calculators, notes, books or other aides are permitted. Scrap paper will be provided.

Be sure to show a few key intermediate steps and make statements in words when deriving results - answers only will not get full marks.

2 Given

The Jacobian determinant of the change of variable $(r, s) \rightarrow (x, t)$ is

$$\frac{\partial(x, t)}{\partial(r, s)} = \det \begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = x_r t_s - x_s t_r = \frac{\partial x}{\partial r} \frac{\partial t}{\partial s} - \frac{\partial x}{\partial s} \frac{\partial t}{\partial r}$$

Total points: 50

3 Question 1

[20 points total]

Suppose you shake the end of a rope of dimensionless length 1 at a certain frequency ω . We neglect gravity and friction and model the waves on the rope using the 1D wave equation:

$$u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0 \quad (1)$$

where u is the displacement of the rope away from its rest state. The rope is attached and held fixed at a wall at $x = 0$,

$$u(0, t) = 0, \quad t > 0. \quad (2)$$

You shake the other end ($x = 1$) sinusoidally, with frequency ω , and give it the displacement

$$u(1, t) = \sin \omega t, \quad t > 0. \quad (3)$$

We assume the rope has zero initial position and velocity

$$u(x, 0) = 0, \quad 0 < x < 1, \quad (4)$$

$$u_t(x, 0) = 0, \quad 0 < x < 1. \quad (5)$$

Note: D'Alembert's solution does not hold for u , because of the BC at $x = 1$.

(a) [10 points] Find a solution of the form

$$U(x, t) = X(x) \sin \omega t \quad (6)$$

that satisfies the PDE (1) and the BCs (2) and (3). (Don't worry about the ICs yet.) Where is the rope stationary (i.e. $U(x, t) = 0$)? For what values of ω is your solution invalid?

(b) [10 points] Use $U(x, t)$ from part (a) to find the full solution $u(x, t)$ to the PDE (1), the BCs (2) and (3), and the ICs (4) and (5). Hint: define a function $v(x, t)$ in terms of $U(x, t)$ and $u(x, t)$ so that $v(x, t)$ satisfies the PDE (1) and has $v(0, t) = 0 = v(1, t)$ (type I zero BCs). Then write down D'Alembert's solution (without derivation) to satisfy the PDE and initial conditions on $v(x, t)$ (don't evaluate the integral in D'Alembert's solution). Then adjust D'Alembert's solution to handle the zero BCs on $v(x, t)$ at $x = 0, 1$. You don't need to evaluate the integral.

4 Question 2

[30 points total]

Consider the following quasi-linear PDE,

$$\frac{\partial u}{\partial t} + (1 + 2u) \frac{\partial u}{\partial x} = -u; \quad u(x, 0) = f(x) \quad (7)$$

where the initial condition is

$$f(x) = \begin{cases} 1, & |x| > 1 \\ 2 - |x|, & |x| \leq 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 2 + x, & -1 \leq x \leq 0 \\ 2 - x, & 0 < x \leq 1 \\ 1, & x > 1 \end{cases}$$

(a) [8 points] By writing the PDE in the form $(A, B, C) \cdot (u_t, u_x, -1) = 0$, where $C = -u$, find the parametric solution using r as your parameter along a characteristic and s to label the characteristic (i.e. the initial value of x). First write down the relevant ODEs for $\partial t/\partial r$, $\partial x/\partial r$, $\partial u/\partial r$. Take the initial conditions $t = 0$ and $x = s$ at $r = 0$. Using the initial condition in (7), write down the IC for u at $r = 0$, in terms of s . Solve for t , u and x (in that order!) as functions of r , s . When integrating for x , be careful: u depends on r !

(b) [8 points] At what time t_{sh} and position x_{sh} does your parametric solution break down? Hint: you might need to consider negative values of $f'(s)$.

(c) [4 points] Write down x in terms of t , s and $f(s)$. For each of $s = -1, 0, 1$, write down x as a function of t .

(d) [4 points] Fill in the table below, using your result from either (a) or (c) to obtain u and x at the s -values listed at time $t = \ln 2$ (note that $e^{-\ln 2} = \frac{1}{2}$, and you may use $\ln 2 = 0.7$):

$s =$	-1	0	1
$u =$			
$x =$			

(e) [6 points] Plot $f(x)$ vs. x . Then, on the SAME plot, plot $u(x, t)$ at $t = \ln 2$ by plotting the three points (x, u) from the table in part (d) and connecting the dots.