The method of characteristics applied to quasi-linear PDEs

18.303 Linear Partial Differential Equations

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1 Motivation

[Oct 26, 2005]

Most of the methods discussed in this course: separation of variables, Fourier Series, Green's functions (later) can only be applied to linear PDEs. However, the method of characteristics can be applied to a form of nonlinear PDE.

1.1 Traffic flow

Ref: Myint-U & Debnath §12.6

Consider the idealized flow of traffic along a one-lane highway. Let $\rho(x,t)$ be the traffic density at (x,t). The total number of cars in $x_1 \leq x \leq x_2$ at time t is

$$N(t) = \int_{x_1}^{x_2} \rho(x,t)\, dx \tag{1}$$

Assume the number of cars is conserved, i.e. no exits. Then the rate of change of the number of cars in $x_1 \leq x \leq x_2$ is given by

$$\begin{aligned} \frac{dN}{dt} &= \text{rate in at } x_1 - \text{ rate out at } x_2 \\ &= \rho(x_1,t)\, V(x_1,t) - \rho(x_2,t)\, V(x_2,t) \\ &= -\int_{x_1}^{x_2} \frac{\partial}{\partial x}(\rho V)\, dx \end{aligned} \tag{2}$$

where $V(x,t)$ is the velocity of the cars at (x,t). Combining (1) and (2) gives

$$\int_{x_1}^{x_2} \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V) \right) dx = 0$$

and since x_1, x_2 are arbitrary, the integrand must be zero at all x,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V) = 0 \tag{3}$$

We assume, for simplicity, that velocity V depends on density ρ, via

$$V(\rho) = c\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

where c = max velocity, $\rho = \rho_{\max}$ indicates a traffic jam ($V = 0$ since everyone is stopped), $\rho = 0$ indicates open road and cars travel at c, the speed limit (yeah right). The PDE (3) becomes

$$\frac{\partial \rho}{\partial t} + c\left(1 - \frac{2\rho}{\rho_{\max}}\right)\frac{\partial \rho}{\partial x} = 0 \tag{4}$$

We introduce the following normalized variables

$$u = \frac{\rho}{\rho_{\max}}, \qquad \tilde{t} = ct$$

into the PDE (4) to obtain (dropping tildes),

$$u_t + (1 - 2u)\, u_x = 0 \tag{5}$$

The PDE (5) is called quasi-linear because it is linear in the derivatives of u. It is NOT linear in $u(x,t)$, though, and this will lead to interesting outcomes.

2 General first-order quasi-linear PDEs

Ref: Guenther & Lee §2.1, Myint-U & Debnath §12.1, 12.2

The general form of quasi-linear PDEs is

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial t} = C \tag{6}$$

where A, B, C are functions of u, x, t. The initial condition $u(x,0)$ is specified at $t = 0$,

$$u(x,0) = f(x) \tag{7}$$

We will convert the PDE to a sequence of ODEs, drastically simplifying its solution. This general technique is known as the method of characteristics and is useful for finding analytic and numerical solutions. To solve the PDE (6), we note that

$$(A, B, C) \cdot (u_x, u_t, -1) = 0. \tag{8}$$

Recall from vector calculus that the normal to the surface $f(x, y, z) = 0$ is ∇f. To make the analogy here, t replaces y, $f(x, t, z) = u(x, t) - z$ and $\nabla f = (u_t, u_x, -1)$. Thus, a plot of $z = u(x, t)$ gives the surface $f(x, t, z) = 0$. The vector $(u_x, u_t, -1)$ is the normal to the solution surface $z = u(x, t)$. From (8), the vector (A, B, C) is the tangent to this solution surface.

The IC $u(x, 0) = f(x)$ is a curve in the $u - x$ plane. For any point on the initial curve, we follow the vector (A, B, C) to generate a curve on the solution surface, called a characteristic curve of the PDE. Once we find all the characteristic curves, we have a complete description of the solution $u(x, t)$.

2.1 Method of characteristics

We represent the characteristic curves parametrically,

$$x = x(r; s), \qquad t = t(r; s), \qquad u = u(r; s),$$

where s labels where we start on the initial curve (i.e. the initial value of x at $t = 0$). The parameter r tells us how far along the characteristic curve. Thus (x, t, u) are now thought of as trajectories parametrized by r and s. The semi-colon indicates that s is a parameter to label different characteristic curves, while r governs the evolution of the solution along a particular characteristic.

From the PDE (8), at each point (x, t), a particular tangent vector to the solution surface $z = u(x, t)$ is

$$(A(x, t, u), B(x, t, u), C(x, t, u)).$$

Given any curve $(x(r; s), t(r; s), u(r; s))$ parametrized by r (s acts as a label only), the tangent vector is

$$\left(\frac{\partial x}{\partial r}, \frac{\partial t}{\partial r}, \frac{\partial u}{\partial r}\right).$$

For a general curve on the surface $z = u(x, t)$, the tangent vector (A, B, C) will be different than the tangent vecto (x_r, t_r, u_r). However, we choose our curves $(x(r; s), t(r; s), u(r; s))$ so that they have tangents equal to (A, B, C),

$$\frac{\partial x}{\partial r} = A, \qquad \frac{\partial t}{\partial r} = B, \qquad \frac{\partial u}{\partial r} = C \tag{9}$$

where (A, B, C) depend on (x, t, u), in general. We have written partial derivatives to denote differentiation with respect to r, since x, t, u are functions of both r and s. However, since only derivatives in r are present in (9), these equations are ODEs! This has greatly simplified our solution method: we have reduced the solution of a PDE to solving a sequence of ODEs.

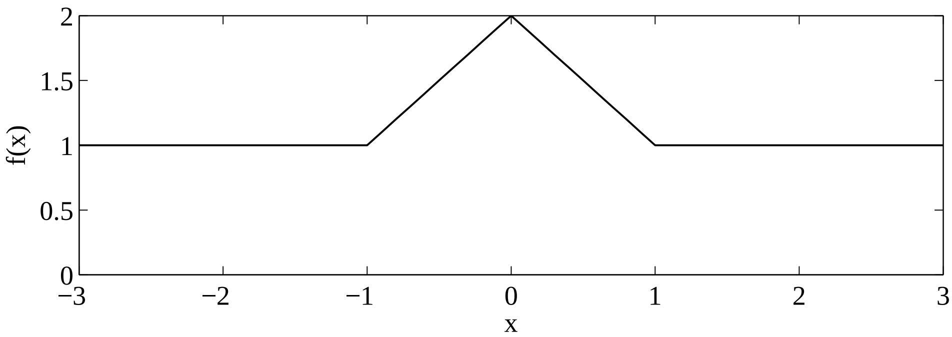

Figure 1: Plot of $f(x)$.

The ODEs (9) in conjunction with some initial conditions specified at $r = 0$. We are free to choose the value of r at $t = 0$; for simplicity we take $r = 0$ at $t = 0$. Thus $t\,(0; s) = 0$. Since x changes with r, we choose s to denote the initial value of $x\,(r; s)$ along the x-axis (when $t = 0$) in the space-time domain. Thus the initial values (at $r = 0$) are

$$x\,(0; s) = s, \qquad t\,(0; s) = 0, \qquad u\,(0; s) = f\,(s)\,. \tag{10}$$

3 Example problem

[Oct 28, 2005]

Consider the following quasi-linear PDE,

$$\frac{\partial u}{\partial t} + (1 + cu)\,\frac{\partial u}{\partial x} = 0, \qquad u\,(x, 0) = f\,(x)$$

where $c = \pm 1$ and the initial condition $f\,(x)$ is

$$f\,(x) = \begin{cases} 1, & |x| > 1 \\ 2 - |x|\,, & |x| \le 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 2 + x, & -1 \le x \le 0 \\ 2 - x, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

The function $f\,(x)$ is sketched in Figure 1. To find the parametric solution, we can write the PDE as

$$(1, 1 + cu, 0) \cdot \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1 \right) = 0$$

Thus the parametric solution is defined by the ODEs

$$\frac{dt}{dr} = 1, \qquad \frac{dx}{dr} = 1 + cu, \qquad \frac{du}{dr} = 0$$

with initial conditions at $r = 0$,

$$t = 0, \qquad x = s, \qquad u = u(x, 0) = u(s, 0) = f(s).$$

Integrating the ODEs and imposing the ICs gives

$$\begin{aligned} t(r; s) &= r \\ u(r; s) &= f(s) \\ x(r; s) &= (1 + cf(s))\, r + s = (1 + cf(s))\, t + s \end{aligned}$$

3.1 Validity of solution and break-down (shock formation)

To find the time t_s and position x_s when and where a shock first forms, we find the Jacobian:

$$\begin{aligned} J &= \frac{\partial(x, t)}{\partial(r, s)} = \det \begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} \\ &= \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r} = 0 - (cf'(s)\, r + 1) = -(cf'(s)\, t + 1) \end{aligned}$$

Shocks occur (the solution breaks down) where $J = 0$, i.e. where

$$t = -\frac{1}{cf'(s)}$$

The first shock occurs at

$$t_s = \min\left(-\frac{1}{cf'(s)}\right)$$

In this course, we will not consider what happens after the shock. You can find more about this in §12.9 of Myint-U & Debnath. We now take cases for $c = \pm 1$.

For $c = 1$, since $\min f'(s) = -1$, we have

$$t_s = -\frac{1}{\min f'(s)} = 1$$

Any of the characteristics where $f'(s) = \min f'(s) = -1$ can be used to find the location of the shock at $t_s = 1$. For e.g., with $s = 1/2$, the location of the shock at $t_s = 1$ is

$$x_s = \left(1 + f\left(\frac{1}{2}\right)\right) 1 + \frac{1}{2} = \left(1 + \left(2 - \frac{1}{2}\right)\right) 1 + \frac{1}{2} = 3.$$

Any other value of s where $f'(s) = -1$ will give the same x_s.

For $c = -1$, since $\max f'(s) = 1$, we have

$$t_s = \frac{1}{\max f'(s)} = 1$$

Any of the characteristics where $f'(s) = \max f'(s) = 1$ can be used to find the location of the shock at $t_s = 1$. For e.g., with $s = -1/2$, the location of the shock at $t_s = 1$ is

$$x_s = \left(1 - f\left(-\frac{1}{2}\right)\right)1 - \frac{1}{2} = \left(1 - \left(2 - \frac{1}{2}\right)\right)1 - \frac{1}{2} = -1.$$

Any other value of s where $f'(s) = 1$ will give the same x_s.

3.2 Solution Method (plotting u(x,t))

Since $r = t$, we can rewrite the solution as being parametrized by time t and the marker s of the initial value of x:

$$x(t; s) = (1 + cf(s))\, t + s, \qquad u(; s) = f(s)$$

We have written $u(; s)$ to make clear that u depends only on the parameter s. In other words, u is constant along characteristics!

To solve for the density u at a fixed time $t = t_0$, we (1) choose values for s, (2) compute $x(t_0; s)$, $u(; s)$ at these s values and (3) plot $u(; s)$ vs. $x(t_0; s)$. Since $f(s)$ is piecewise linear in s (i.e. composed of lines), x is therefore piecewise linear in s, and hence at any given time, $u = f(s)$ is piecewise linear in x. Thus, to find the solution, we just need to follow the positions of the intersections of the lines in $f(s)$ (labeled by $s = -1, 0, 1$) in time. We then plot the positions of these intersections along with their corresponding u value in the u vs. x plane and connect the dots to obtain a plot of $u(x, t)$. Note that for $c = 1$, the $s = -1$, 0, 1 characteristics are given by

$$\begin{aligned} s &= -1 : x = (1 + cf(-1))\, t - 1 = 2t - 1 \\ s &= 0 : x = (1 + cf(0))\, t + 0 = 3t \\ s &= 1 : x = (1 + cf(1))\, t + 1 = 2t + 1 \end{aligned}$$

These are plotted in Figure 2. The following tables are useful as a plotting aid:

$t = \frac{1}{2}$

$s =$	-1	0	1
$u =$	1	2	1
$x =$	0	$\frac{3}{2}$	2

$t = t_s = 1$

$s =$	-1	0	1
$u =$	1	2	1
$x =$	1	3	3

A plot of $u(x, 1/2)$ is made by plotting the three points (x, u) from the table for $t = 1/2$ and connecting the dots (see middle plot in Figure 3). Similarly, $u(x, t_s) = u(x, 1)$ is plotted in the last plot of Figure 3.

Repeating the above steps for $c = -1$, the $s = -1$, 0, 1 characteristics are given by

$$\begin{aligned} s &= -1 : x = (1 - f(-1))\, t - 1 = -1 \\ s &= 0 : x = (1 - f(0))\, t + 0 = -t \\ s &= 1 : x = (1 - f(1))\, t + 1 = 1 \end{aligned}$$

These are plotted in Figure 4. We then construct the tables:

$t = \frac{1}{2}$

$s =$	-1	0	1
$u =$	1	2	1
$x =$	-1	$-\frac{1}{2}$	1

$t = t_s = 1$

$s =$	-1	0	1
$u =$	1	2	1
$x =$	-1	-1	1

As before, plots of $u(x, 1/2)$ and $u(x, 1)$ are made by plotting the three points (x, u) from the tables and connecting the dots. See middle and bottom plots in Figure 5. Note that for $c = 1$ the wave front steepened, while for $c = -1$ the wave tail steepened. This is easy to understand by noting how the speed changes relative to the height u of the wave. When $c = 1$, the local wave speed $1 + u$ is larger for higher parts of the wave. Hence the crest catches up with the trough ahead of it, and the shock forms on the front of the wave. When $c = -1$, the local wave speed $1 - u$ is larger for higher parts of the wave; hence the tail catches up with the crest, and the shock forms on the back of the wave.

4 Solution to traffic flow problem

[Oct 31, 2005]

The traffic flow PDE (5) is

$$u_t + (1 - 2u)\, u_x = 0 \tag{11}$$

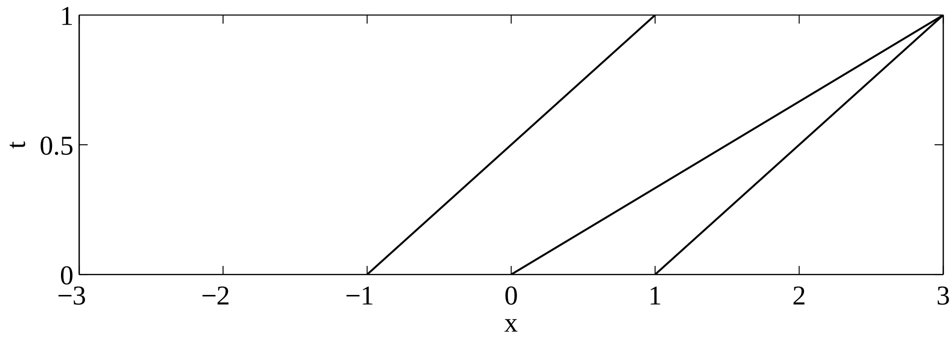

Figure 2: Plot of characteristics for $c = 1$.

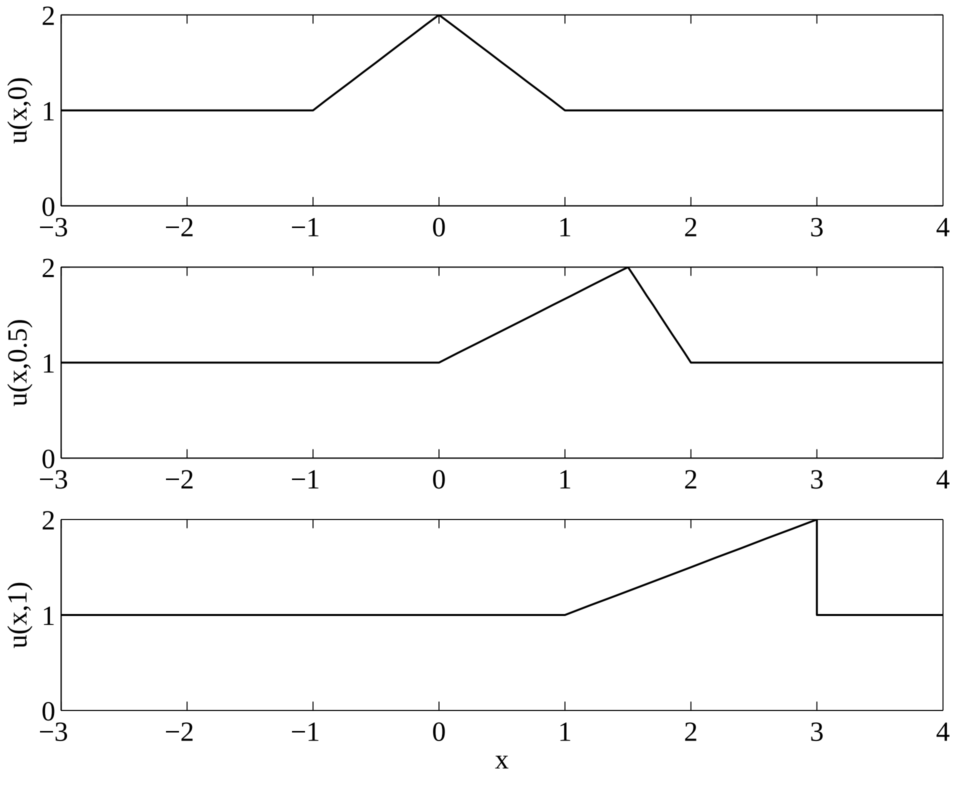

Figure 3: Plot of $u(x, t_0)$ with $c = 1$ for $t_0 = 0$, 0.5 and 1.

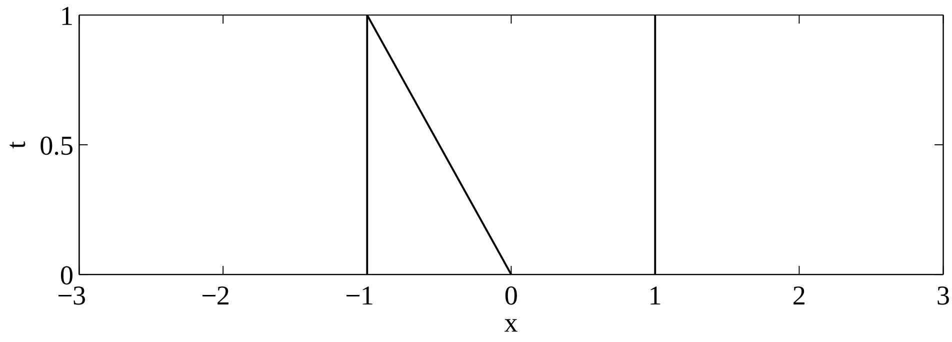

Figure 4: Plot of characteristics with $c = -1$.

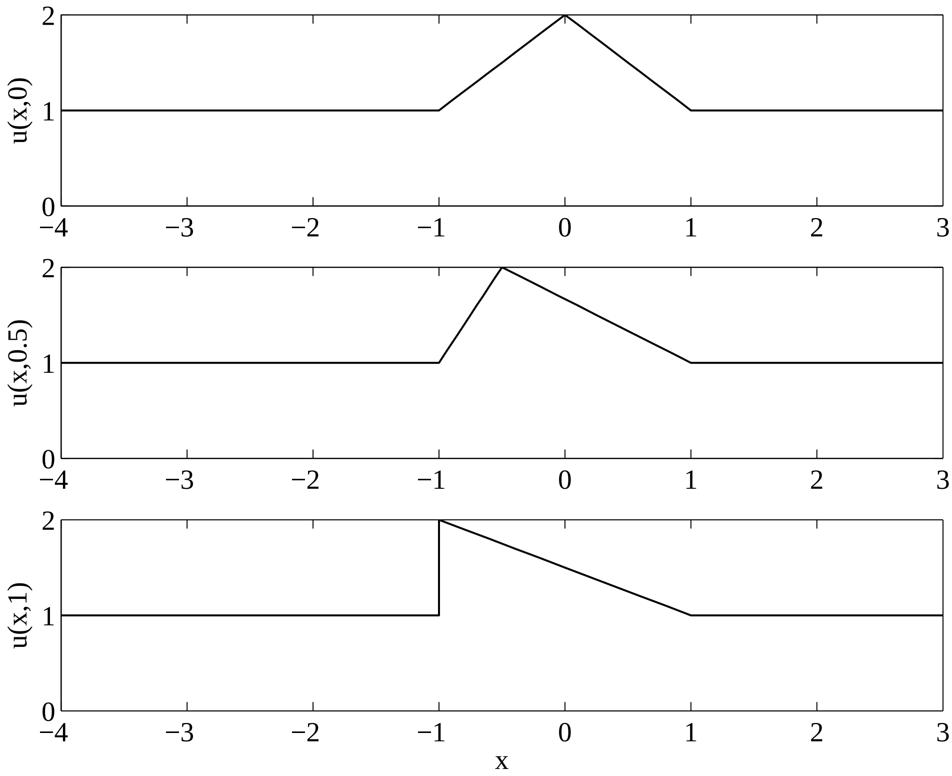

Figure 5: Plot of $u(x, t_0)$ with $c = -1$ for $t_0 = 0$, 0.5 and 1.

and has form (6) with $(A, B, C) = (1 - 2u, 1, 0)$. The characteristic curves satisfy (9) and (10)

$$\begin{aligned}
\frac{\partial x}{\partial r} &= 1 - 2u, \qquad && x(0) = s, \\
\frac{\partial t}{\partial r} &= 1, && t(0) = 0, \\
\frac{\partial u}{\partial r} &= 0, && u(0) = f(s).
\end{aligned}$$

Integrating gives the parametric equations

$$t = r + c_1, \qquad u = c_2, \qquad x = (1 - 2u) r + c_3 = (1 - 2c_2) r + c_3$$

Imposing the ICs gives $c_1 = 0$, $c_2 = f(s)$, $c_3 = s$, so that

$$t = r, \qquad u = f(s), \qquad x = (1 - 2f(s)) r + s = (1 - 2f(s)) t + s \tag{12}$$

We can now write

$$x(t; s) = (1 - 2f(s)) t + s, \qquad u(; s) = f(s)$$

Again, the traffic density u is constant along characteristics. Note that this would change if, for example, there was a source/sink term in the traffic flow equation (11), i.e.

$$u_t + (1 - 2u) u_x = h(x, t, u)$$

where $h(x, t, u)$ models the traffic loss / gain to exists and on-ramps at various positions.

4.1 Example : Light traffic heading into heavier traffic

Consider light traffic heading into heavy traffic, and model the initial density as

$$u(x, 0) = f(x) = \begin{cases} \alpha, & x \le 0 \\ \left(\frac{3}{4} - \alpha\right) x + \alpha, & 0 \le x \le 1 \\ \frac{3}{4}, & x \ge 1 \end{cases} \tag{13}$$

where $0 \le \alpha \le 3/4$. The lightness of traffic is parametrized by α. We consider the case of light traffic $\alpha = 1/6$ and moderate traffic $\alpha = 1/3$.

From (12), the characteristics are [DRAW]

$$x = \begin{cases} (1 - 2\alpha) t + s, & s \le 0 \\ (1 - 2\alpha - 2(3/4 - \alpha) s) t + s, & 0 \le s \le 1 \\ -t/2 + s, & s \ge 1 \end{cases}$$

For $\alpha = 1/6$, we have

$$x = \begin{cases} \frac{2}{3}t + s, & s \le 0 \\ \left(\frac{2}{3} - \frac{7}{6}s\right)t + s, & 0 \le s \le 1 \\ -\frac{1}{2}t + s, & s \ge 1 \end{cases}$$

For $\alpha = 1/3$, we have

$$x = \begin{cases} \frac{1}{3}t + s, & s \le 0 \\ \left(\frac{1}{3} - \frac{5}{6}s\right)t + s, & 0 \le s \le 1 \\ -\frac{1}{2}t + s, & s \ge 1 \end{cases}$$

Again, for fixed times $t = t_0$, plotting the solution amounts to choosing an appropriate range of values for s, in this case $-2 \le s \le 2$ would suffice, and then plotting the resulting points $u(t_0, s)$ versus $x(t_0, s)$ in the xu-plane.

The transformation $(r, s) \rightarrow (x, t)$ is non-invertible if the determinant of the Jacobian matrix is zero,

$$\frac{\partial(x,t)}{\partial(r,s)} = \det\begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = \det\begin{pmatrix} 1 - 2f(s) & -2f'(s)r + 1 \\ 1 & 0 \end{pmatrix} = 2f'(s)r - 1 = 0. \tag{14}$$

Solving for r and noting that $t = r$ gives the time when the determinant becomes zero,

$$t = r = \frac{1}{2f'(s)}. \tag{15}$$

Since times in this problem are positive $t > 0$, then shocks occur if $f'(s) > 0$ for some s. The first such time where shocks occur is

$$t_{shock} = \frac{1}{2\max\{f'(s)\}}. \tag{16}$$

In the example above, the time when a shock first occurs is given by substituting (13) into (16),

$$t_{shock} = \frac{1}{2\max\{f'(s)\}} = \frac{1}{2\left(\frac{3}{4} - \alpha\right)}.$$

Thus, lighter traffic (smaller α) leads to shocks sooner! The position of the shock at t_{shock} is given by

$$x_{shock} = (1 - 2\alpha)\,t_{shock} = \frac{\frac{1}{2} - \alpha}{\frac{3}{4} - \alpha}.$$