18.307: Integral Equations:

NAME: Solutions to Quiz 1 - Fall 2003

Pick-up Time:

Return Time:

18.307 Take–Home Quiz # 1

Thursday, October 16, 2003

You are given 5 hours to do the test.

You are NOT allowed to communicate with anyone, other than the instructor, about this quiz before 12 midnight today. Work on the test by yourself!

You can use any books or notes, but you are NOT allowed to use calculators, or any software such as Mathematica or Maple for example, in order to solve the problems.

Read the problems CAREFULLY. Justify your answers. Cross out what is not meant to be part of your final answer. Total # of points: 120.

I. (25 pts) Find the λ for which the integral equation (IE)

$$v(x) = \lambda \int_0^\infty dy\, e^{-2y}\,(y + e^x)\, v(y)$$

has non-trivial solutions. Give the corresponding $v(x)$.

Solution : Let

$$\alpha = \int_0^\infty dy\; e^{-2y}\, y\, v(y)\,, \qquad \beta = \int_0^\infty dy\; e^{-2y}\, v(y).$$

Then,

$$v(x) = \lambda(\alpha + \beta x)$$

By substitution in the equations for α and β we find

$$\alpha = \lambda \int_0^\infty dy\; e^{-2y}.y\,(\alpha + \beta e^y) = \lambda\left(\frac{\alpha}{4} + \beta\right)$$

$$\beta = \lambda \int_0^\infty dy\; e^{-2y}\,(\alpha + \beta e^y) = \lambda\left(\frac{\alpha}{2} + \beta\right)$$

So, the system of equations for α and β is :

$$\begin{cases} \left(1 - \frac{\lambda}{4}\right)\alpha - \lambda\beta = 0 \\ -\frac{\lambda}{2}\alpha + (1-\lambda)\beta = 0 \end{cases}, \quad \text{with non-trivial solutions if}$$

$$\begin{vmatrix} 1-\lambda/4 & -\lambda \\ -\lambda/2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \left(1-\frac{\lambda}{4}\right)(1-\lambda) - \lambda\cdot\lambda/2 = 0 \Rightarrow \lambda^2 + 5\lambda - 4 = 0 \Rightarrow \lambda = \frac{-5 \pm \sqrt{41}}{2}$$

The corresponding (α, β) are found by either of the two equations above.

$$\alpha = -2\,\frac{\lambda - 1}{\lambda}\,\beta = \begin{cases} \dfrac{\sqrt{41}-3}{4}\,\beta & \text{for } \lambda = \dfrac{-5+\sqrt{41}}{2} \\[2ex] -\dfrac{\sqrt{41}+3}{4}\,\beta & \text{for } \lambda = \dfrac{-5-\sqrt{41}}{2} \end{cases}$$

The corresponding solution is : $\quad v = \underbrace{\dfrac{-5 \pm \sqrt{41}}{2}}_{\lambda} \cdot \beta\left(\dfrac{-3 \pm \sqrt{41}}{4} + x\right)$, β: arbitrary $\neq 0$

II.(35 pts) The concentration $c = c(x,t)$ on the surface of a material obeys the partial differential equation (PDE)

$$\frac{\partial c}{\partial t} + \alpha \frac{\partial^4 c}{\partial x^4} = F(c), \quad t > 0, \quad -\infty < x < \infty, \quad \alpha > 0 : \text{const.},$$

and the initial condition $c(x,0) = g(x)$, where $g(x)$ is given, while $c(x,t)$ and its derivatives approach 0 as $|x| \to \infty$; $F(c)$ is a given bounded, nonlinear function of $c = c(x,t)$ which arises from the concentration-dependent deposition of material on the surface.

Find a suitable (causal) Green's function $G(x,t;x',t')$ for this problem, along with the corresponding solution c_0 of the homogeneous PDE; simplify the final expression for G as much as possible. Convert the given PDE to an IE; give this IE.

Hint: If you encounter any complicated integral(s) for G, try to express your results in terms of single (one-dimensional) integral(s), or in simpler form if possible.

Solution: We assume that $F(0)=0$ so that there is no deposition of material under conditions of zero c.

The concentration $c(x,t)$ is written as

$$c(x,t) = c_0(x,t) + c_p(x,t) \tag{1}$$

where $c_0(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial c_0}{\partial t} + \alpha \dfrac{\partial^4 c_0}{\partial x^4} = 0 \\ c_0(x,0) = g(x) \\ |c_0| \to 0 \text{ as } |x| \to \infty; \text{ same for derivatives of } c_0, \end{cases} \tag{I}$$

and $c_p(x,t)$ is a particular solution given in terms of the Green function

$G(x,t;x',t')$:

$$c_p(x,t) = \int_{-\infty}^{\infty} dx' \int_0^{\infty} dt' \; G(x,t;x',t') \; F(c(x',t')) \,, \qquad -\infty < x < \infty, \; t > 0, \qquad (2)$$

where we take $c=0$ for $t<0$. The Green function satisfies

$$\begin{cases} \dfrac{\partial G}{\partial t} + \alpha \dfrac{\partial^4 G}{\partial x^4} = \delta(x-x') \cdot \delta(t-t') \,, \quad -\infty < x, t < \infty \\ G(x,0;x',t'>0) = 0, \; G \to 0 \text{ as } |x| \to 0 \text{; same for derivatives in } x \end{cases} \qquad (\mathrm{II})$$

- We find c_0 by solving problem (I) first.

(from PDE for c_0)

FT: $c_0(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{+ikx} \tilde{c}_0(k,t) \Rightarrow \frac{\partial \tilde{c}_0}{\partial t} + \alpha k^4 \tilde{c}_0 = 0 \Rightarrow \tilde{c}_0(k,t) = A(k)\, e^{-\alpha k^4 t}$

From $c_0(x,0) = g(x)$, we get $A(k) = \int_{-\infty}^{\infty} dx \; g(x)\, e^{-ikx}$: FT of $g(x)$

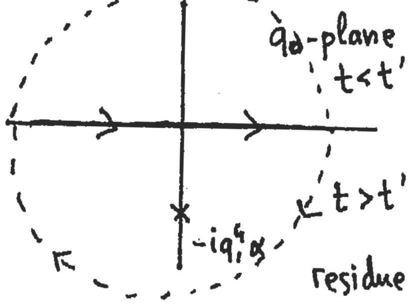
Hence, $c_0(x,t)$ is considered as known: $c_0(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} e^{-\alpha k^4 t} A(k)$. (3)

- Next, we find $G(x,t;x',t')$ by solving problem (II) :

Take FT in both x and t: $G(x,t;x',t') = \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \; e^{iq_1(x-x') - iq_0(t-t')} \cdot \tilde{G}(q_0,q_1)$

$\Rightarrow (-iq_0 + q_1^4 \alpha)\, \tilde{G} = 1$ from PDE for G.

$\therefore \; \tilde{G}(q_0,q_1) = \dfrac{1}{q_1^4 \alpha - iq_0}$, which has a simple pole at $q_0 = -iq_1^4 \alpha$.



$\underline{t<t'}$: Path closes in upper half q_0-plane, $G=0$.

$\underline{t>t'}$: Path closes in lower half plane, and picks up residue at $q_0 = -iq_1^4\alpha$: $G(x,t;x',t') = \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \; e^{iq_1(x-x') - q_1^4 \alpha (t-t')}$: cannot be further simplified

The desired IE follows then from (1) with (2) and (3).

III. (35 pts) Consider the integral equation (IE)

$$u(x) = x + \frac{1}{2}\int_0^A dy\,|y-x|\,u(y), \quad 0 < x < A.$$

(a) (20 pts). Solve the IE for $A = 1$.

(b) (15 pts) Does the given IE have a solution for $A = \infty$? Explain.

Hint: Beware: Since the lower limit of integration is 0 (or any finite number for that purpose), the application of Fourier transform is questionable. If you decide to "mutilate" the IE in any way, don't forget to get back to it when needed. In (b) you are NOT asked to solve the IE.

Solution: (a) The IE reads:

$$u(x) = x + \frac{1}{2}\int_0^x dy\,(x-y)\,u(y) + \frac{1}{2}\int_x^1 dy\,(y-x)\,u(y), \quad 0<x<1. \qquad (1)$$

$\Rightarrow$ Differentiate once:

$$u'(x) = 1 + \frac{1}{2}\int_0^x dy\,u(y) - \frac{1}{2}\int_x^1 dy\,u(y) \qquad (2)$$

Once more: $u''(x) = \frac{1}{2}u(x) + \frac{1}{2}u(x) = u(x)$ $\therefore$ $u''(x) - u(x) = 0$

Solution: $u(x) = \bar{C}_1 e^{x} + \bar{C}_2 e^{-x} = C_1 \sinh x + C_2 \cosh x$

Conditions for $u(x)$ are found from (1) and (2):

Eq. (2): $\begin{cases} u'(0) = 1 - \frac{1}{2}\int_0^1 dy\,u(y) \\ u'(1) = 1 + \frac{1}{2}\int_0^1 dy\,u(y) \end{cases}$ $\therefore$ $u'(0) + u'(1) = 2 \qquad (3)$

Eq. (1): $\begin{cases} u(0) = \frac{1}{2}\int_0^1 dy\,y\,u(y) \\ u(1) = 1 + \frac{1}{2}\int_0^1 dy\,u(y) - \frac{1}{2}\int_0^1 dy\cdot y\,u(y) = 1 - \frac{1}{2}[u'(0)-u'(1)] - \frac{1}{2}\int_0^1 dy\cdot y\,u(y) \end{cases}$

(4)

$\therefore$ $u(0) + u(1) = 1 - \frac{1}{2}[u'(0) - u'(1)]$

Hence, we solve the system of (3), (4) to get C_1 and C_2:

$$\begin{cases} C_1\,(1+\cosh 1) + C_2 \sinh 1 = 2 \\ C_1\cdot(\sinh 1 + \frac{1}{2} - \frac{1}{2}\cosh 1) + C_2\,(1+\cosh 1 - \frac{1}{2}\sinh 1) = 1 \end{cases}$$

$$\therefore \quad C_1 = \frac{2+2\cosh 1 - 2\sinh 1}{2+2\cosh 1 - \sinh 1}, \quad C_2 = \frac{2\cosh 1 - 2\sinh 1}{2+2\cosh 1 - \sinh 1}$$

(b) We can show that the IE has NO solution when $A=\infty$.

Roughly, in this case one has to impose the condition that the integral converges:

$$\left| \int_0^\infty dy\ |x-y|\cdot u(y) \right| < \infty \tag{5}$$

The IE still reduces to the ODE $u''-u=0$ with solution

$$u(x) = \bar{C}_1 e^{+x} + \bar{C}_2 e^{-x}.$$

By (5) we get $\boxed{\bar{C}_1 = 0}$. $\therefore$ $u(x) = \bar{C}_2 e^{-x}$; it remains to find $\bar{C}_2$.

Let's take the limit $x\to+\infty$ (or x very large) in the original IE:

$$u(x) \simeq x + \frac{1}{2}\int_0^\infty dy\ (x-y)\ u(y), \qquad x\to+\infty$$

$$= x\left[1+\frac{1}{2}\int_0^\infty dy\ u(y)\right] - \frac{1}{2}\int_0^\infty dy\cdot y\, u(y), \qquad x\to+\infty$$

which has to approach 0 since $u(x) = C_2\cdot e^{-x} \to 0$ as $x\to+\infty$.

It follows that $1+\frac{1}{2}\int_0^\infty dy\ u(y) = 0 \Rightarrow 1+\frac{C_2}{2} = 0 \Rightarrow C_2 = -2$

and $\int_0^\infty dy\cdot y\, u(y) = 0$: impossible to satisfy. Hence, the IE has NO solution for $A=+\infty$.

IV.(25 pts) (a)(18 pts) <u>Solve</u> entirely the following system of integral equations:

$$u_1(x) = u_2(x) + \lambda \int_0^x dy\, u_1(x-y)\, u_2(y) \qquad (1)$$

$$u_2(x) = 1 + \mu \int_0^x dy\, u_2(x-y)\, u_1(y), \qquad (2)$$

where $0 < x < \infty, \lambda$ and μ are given constants, and $u_1(x)$ and $u_2(x)$ are <u>both unknown</u> functions to be found, <u>without</u> using any iteration or recursion scheme.

(b)(7 pts) <u>Discuss</u> at least <u>two</u> distinct ways by which one can actually evaluate $u_1(x)$ and $u_2(x)$ as power series in x. <u>One</u> of these ways <u>must</u> follow from your solution of part (a).

Remark: In part (b) you are NOT asked to give the power series for u_1 and u_2 but just discuss clearly how one can get them from the solution found in (a), or by other means. In (b) you are allowed to use any alternative scheme you deem suitable.

<u>Solution</u>: (a) We notice that both integrals express convolutions. It is thus convenient to use Laplace transform:

$$\bar{u}_1(s) = \int_0^\infty dx\; u_1(x)\, e^{-sx}, \qquad \bar{u}_2(x) = \int_0^\infty dx\; u_2(x)\, e^{-sx}$$

The system of IE is thus converted to a system of algebraic equations:

(1): $\bar{u}_1(s) = \bar{u}_2(s) + \lambda\, \bar{u}_1(s) \cdot \bar{u}_2(s) \Rightarrow \bar{u}_1(s) = \dfrac{\bar{u}_2(s)}{1-\lambda \cdot \bar{u}_2(s)}$

(2): $\bar{u}_2(s) = \dfrac{1}{s} + \mu\, \bar{u}_2(s) \cdot \bar{u}_1(s) \Rightarrow \bar{u}_2(s) = \dfrac{\frac{\lambda}{s}+1 \pm \sqrt{\left(1+\frac{\lambda}{s}\right)^2 - 4\,\frac{\mu+\lambda}{s}}}{2(\mu+\lambda)}$

We choose the sign so that $\bar{u}_2(s) \longrightarrow 0$ as $s \to \infty$:

For $s \to \infty$, $\sqrt{\left(1+\frac{\lambda}{s}\right)^2 - 4\,\frac{\mu+\lambda}{s}} \approx 1 - \frac{2\mu+\lambda}{s}$, $\bar{u}_2(s) \approx \dfrac{\frac{\lambda}{s}+1 \pm \left(1 - \frac{2\mu+\lambda}{s}\right)}{2(\mu+\lambda)} \longrightarrow 0$ if '$-$' is chosen.

It follows that the <u>lower</u> $(-)$ sign must be chosen:

$$\therefore \quad \bar{u}_2(s) = \frac{\frac{\lambda}{s} + 1 - \sqrt{\left(1+\frac{\lambda}{s}\right)^2 - 4\,\frac{\mu+\lambda}{s}}}{2(\mu+\lambda)} .$$

Then, $\bar{u}_1(s)$ is given by

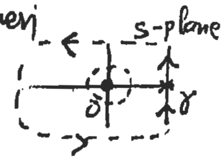
$$\bar{u}_1(s) = \frac{\dfrac{\frac{\lambda}{s} + 1 - \sqrt{\left(\frac{\lambda}{s}+1\right)^2 - 4\,\frac{\mu+\lambda}{s}}}{2(\mu+\lambda)}}{1 - \lambda\,\dfrac{\frac{\lambda}{s}+1-\sqrt{\cancel{4} - 4\,\frac{\mu+\lambda}{s} + \left(1+\frac{\lambda}{s}\right)^2}}{2(\mu+\lambda)}} = \frac{\frac{\lambda}{s} + 1 - \sqrt{\cancel{4} - 4\,\frac{\mu+\lambda}{s} + \left(1+\frac{\lambda}{s}\right)^2}}{-\lambda + 2(\mu+\lambda) - \frac{\lambda^2}{s} + \lambda\sqrt{1 - 4\,\frac{\mu+\lambda}{s}}}$$

Then $u_1(x)$ and $u_2(x)$ follow by inversion

$$u_1(x) = \frac{1}{2\pi i}\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} ds\; e^{sx}\,\bar{u}_1(s), \qquad u_2(x) = \frac{1}{2\pi i}\int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} ds\; e^{sx}\,\bar{u}_2(s) \qquad (3)$$

where the paths lie to the <u>right</u> of all singularities of the integrands.

(b) <u>Method A</u>: Shift the paths in (3) to the right by forcing $|s|$ to be "large". Then the integrands can be expanded in Taylor series of $1/s$. By interchanging summation and integration one then encounters integrals of form



$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} ds\,\frac{e^{sx}}{s^n} = \operatorname*{Res}_{s=0}\left\{\frac{e^{sx}}{s^n}\right\} = \frac{x^{n-1}}{(n-1)!}$$

with an integrand which has a n^{th}-order pole at $s=0$, and <u>no</u> other singularity.

Recall: The residue is the coefficient of $\frac{1}{s}$ in the Laurent series of s; by expanding $e^{sx} = 1 + sx + \dots + \frac{(sx)^{n-1}}{(n-1)!} + \dots$ we notice that only $\frac{(sx)^{n-1}}{(n-1)!}$ gives the $\frac{1}{s}$ with the s^n of he denominator.]

<u>Method B</u>: Apply <u>iteration</u> by originally setting $\lambda = \mu = 0 \overset{(1),(2)}{\Rightarrow} u_1(x) = u_2(x) = 1$ hen replace u_1 and u_2 under the integral signs by these expressions. etc. Because he integrals are of form $\int_0^x$ the procedure will produce power series in x.

Iteration procedure:

Step 0 $(\lambda=\mu=0)$: $u_1^{(0)}(x) = u_2^{(0)}(x) = 1$

Step 1 :
$$\left.\begin{aligned} u_1^{(1)}(x) &= u_2^{(1)}(x) + \lambda \int_0^x dy\; u_1^{(0)}(x-y)\, u_2^{(0)}(y) = u_2^{(1)} + \lambda x \\ u_2^{(1)}(x) &= 1 + \mu \int_0^x dy\; u_2^{(0)}(x-y)\, u_1^{(0)}(y) = 1 + \mu x \end{aligned}\right\}$$

$$\Rightarrow \begin{cases} u_1^{(1)}(x) = 1 + (\lambda+\mu)x \\ u_2^{(1)}(x) = 1 + \mu x \end{cases}$$

Step 2 :
$$u_1^{(2)}(x) = u_2^{(2)}(x) + \lambda \int_0^x dy\; u_1^{(1)}(x-y)\; u_2^{(1)}(y)$$
$$u_2^{(2)}(x) = 1 + \mu \int_0^x dy\; u_2^{(1)}(x-y)\; u_1^{(1)}(y)$$

$$\vdots$$

Step (n) :
$$u_1^{(n)}(x) = u_2^{(n)}(x) + \lambda \int_0^x dy\; u_1^{(n-1)}(x-y)\; u_2^{(n-1)}(y)$$
$$u_2^{(n)}(x) = 1 + \mu \int_0^x dy\; u_2^{(n-1)}(x-y)\; u_1^{(n-1)}(y)$$

where $u_i^{(n)}$ denotes the nth iterate of $u_i(x)$.

This procedure yields a power series in x.