18.311: Principles of Applied Mathematics — Rodolfo Rosales
Lecture 33 — Spring 2014

Separation of Variables and Normal Modes

Another way to solve the wave equation (works for other equations too).

Example: do heat equation in 1D with T=0 at ends.
do wave equation in 1D with $u_x = 0$ at ends.

Note: for the wave equation the solutions obtained in this way should be compatible with the form u = f(x-c*t) + g(x+c*t).

Exercise:
Typical separation of variables solution has the form:

u = cos(n*π*t/L)*sin(n*π*x/L)

which yields

u = (1/2)*cos((n*π*/L)*(x+t)) + (1/2)*cos((n*π*/L)*(x-t))

using trigonometric equalities.

Normal modes. Equations of the form $u_t = Lu$
Relationship with separation of variables: equation invariant under time shift allows separation u = exp(λ*t) U(x)

Example: write wave equation as $u_t = v$ and $v_t = u_{xx}$.
heat equation: $u_t = u_{xx}$

- Note analogy with linear o.d.e. dY/dt = A*Y, A NxN matrix, solved by finding eigenvalues and eigenvectors of A.
- Hence look for solutions of the form u = e^{\λ*t} v(x). Solve and find normal modes (eigenvalues and eigenfunctions).

General solution: Superposition ... leads to Fourier Series, etc.

Example: heat equation with various B.C.

1. In a ring: periodic.
2. Zero T at ends.
3. Zero flux at ends.

Various types of Fourier series.

Explain this works, for example, as long as the assocated eigenvalue problem is self-adjoint:

1) Interpretation of matrices as linear operators.
2) Interpretation of self-adjoint for matrices in terms of the scalar product.
3) Definition of scalar product and Hilbert space.

Consider a string tied at the ends. Use a-dimensional variables. Then:

$u_{tt} - u_{xx} = 0$ and $u = 0$ at $x = 0$ and $x = L$

Find normal modes (or separate variables), and find connection with characteristics: Normal modes as superpositions of a right and a left traveling wave.

Wave equation. Show that:
Boundary conditions for a tied string of length L lead to a solution of space period P = 2*L --- extend solution "reflecting" across ends.

Example of normal modes.
$T_t = T_{xx}$ for $0 < x < 1$, with:
$T = 0$ for $x = 0$, and [E]
$T_x + T = 0$ for $x = 1$.

Physical meaning of the boundary conditions (heat and elasticity)

- Dirichlet: ice bath or rigid clamped end.
- Neumann: flux prescribed or free end (no stress).
- Robin: fluid cooling or elastically clamped end.

Show space operator in [E] is self-adjoint and negative ==> eigenvalues real and negative.
Calculate eigenvalues and use to write solution in terms of the initial value T(x, 0) = f(x).
Graphical solution of the equation for the eigenvalues: $\lambda = - k^2$, where $k*\cos(k) + \sin(k) = 0$, and $k > 0$.
Plot k versus tan(k) and show solutions k_n, $k_n \sim \pi*(n-1/2)$, n = 1, 2, ...
Explain how to solve using Newton's method.

Brief description of separation of variables, and do example $u_{xx} + u_{yy} = 0$ for $r < 1$ and u given on r = 1.
Use polar coordinates, so $r(r*u_r)$_r + u_{theta} theta = 0.
Point out method works only for some equations in some coordinate systems.

Students should read the "short notes on separation of variables" included with problem set #7.

Normal modes. Example of heat equation in $0 < x < \pi$, with zero BC.

Linear algebra review: begin with self-adjoint and scalar products.

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