18.314 PRACTICE FINAL EXAM

(for Final Exam of December 15, 2014)

Closed book, calculators, computers, iPods, cell phones, etc., but you may bring in one sheet of paper, no larger than $8\frac{1}{2}'' \times 11''$, written on both sides. Do all eight problems. There are a total of 80 points. Be sure to **show your work** on each problem.

1. (a) (5 points) Solve the recurrence $f(0) = 2$, $f(1) = 4$, and

$$f(n+2) = 4f(n+1) - 2f(n), \quad n \geq 0.$$

(Give a simple explicit formula for $f(n)$. It is o.k. to use irrational numbers.)

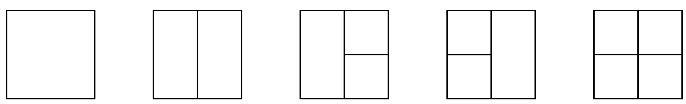
(b) (5 points) Show that $\lfloor (2+\sqrt{2})^n \rfloor$ is odd for all integers $n \geq 0$.

2. (10 points) Let $f(n)$ be the number of ways to choose a permutation π of $1, 2, \ldots, n$ and color each cycle of π of even length either red or blue. For instance, $f(1) = 1$, $f(2) = 3$, $f(3) = 9$, and $f(4) = 45$. Set $f(0) = 1$. Find the generating function

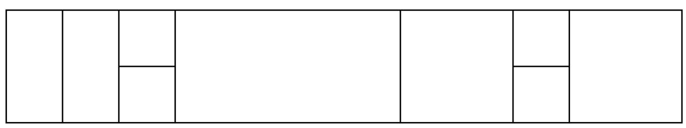
$$F(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!}.$$

To get full credit, your answer should not involve any infinite sums, infinite products, or the functions exp (the exponential function) and log.

3. (a) (5 points) Let $f(n)$ be the number of ways to tile a $2 \times n$ rectangle with 1×1 squares and $2 \times k$ rectangles for any integer $k \geq 1$, where the 2×1 rectangle must be vertical. (Set $f(0) = 1$.) For instance, $f(2) = 5$, given by



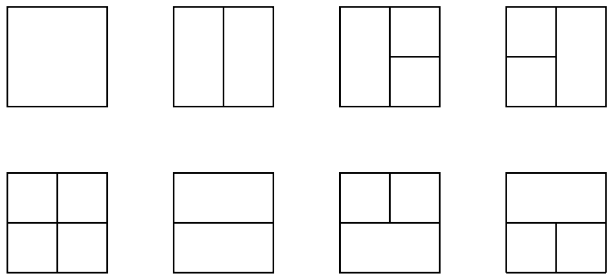
A larger example of such a tiling is given by



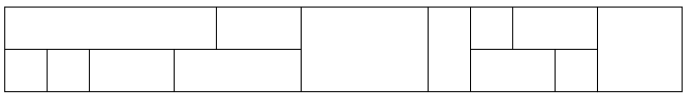
Find a simple expression for the generating function

$$F(x) = \sum_{n \geq 0} f(n) x^n.$$

(b) (5 points; difficult) Let $g(n)$ be the number of ways to tile a $2 \times n$ rectangle with $a \times b$ rectangles for any integers $a, b \geq 1$. (Set $g(0) = 1$.) For instance, $g(2) = 8$, given by



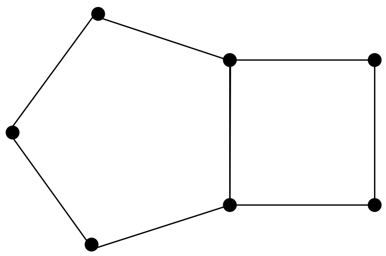
A larger example of such a tiling is given by



Find a simple expression for the generating function

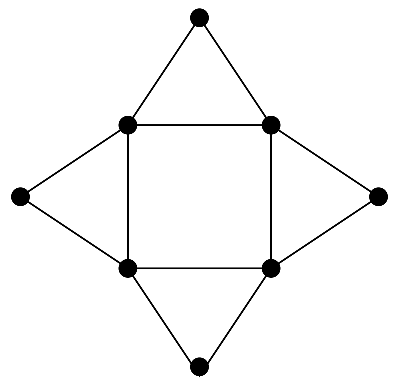
$$G(x) = \sum_{n \geq 0} g(n) x^n.$$

4. (10 points) Let $m, n \geq 3$. Let G be the graph obtained by identifying an edge of an m-cycle with the edge of an n-cycle. Thus G has $m+n-2$ vertices and $m+n-1$ edges. Find the number $\kappa(G)$ of spanning trees of G. (It is easiest to use "naive" reasoning and not to use the Matrix-Tree Theorem.) The figure below shows the case $m = 5$ and $n = 4$.



5. (10 points) Let G be a regular bipartite graph of degree $d \geq 2$, i.e., every vertex of G has the same degree $d \geq 2$. Show that G has a spanning subgraph (that is, a subgraph using every vertex of G) that is a disjoint union of cycles. (No two cycles should have a vertex in common.)

6. (10 points) Compute the chromatic polynomial of the following graph G with eight vertices:



7. (a) (5 points) Does there exist a planarly embedded graph with no isthmus (i.e., no edge e for which the same face lies on both sides of e) and with exactly one face with k vertices, $3 \leq k \leq 8$, and with no other faces? Thus G has six faces in all.

   (b) (5 points) Same question, but for $3 \leq k \leq 9$.

8. (10 points) Find the least positive integer n with the following property: if the edges of K_n are colored red and blue, then there must exist a monochromatic non-closed path of length three, i.e., a path of length three (not a triangle, so with four vertices and three edges) such that all edges in the path have the same color. (Five points for showing that *some* value of n has this property. Full credit for finding the *least* such value and showing that it is indeed least.)

18.314 Combinatorial Analysis
Fall 2014