HOMEWORK 3 (18.315, FALL 2005)

1) Decide whether a rectangle $[50 \times 60]$ can be tiles with rectangles

$a)$ $[20 \times 15]$ $b)$ $[5 \times 8]$

$c)$ $[6.25 \times 15]$ $d)$ $[2 - \sqrt{2} \times 2 + \sqrt{2}]$

$e)$ Find and prove a general criterion for tileability of a rectangle $[a \times b]$ with rectangular tiles $[p \times q]$.

2) Let u_n be the number of *alternating permutations* $\sigma \in S_n$, i.e. permutations with $\sigma(1) < \sigma(2) > \sigma(3) < \ldots$ Prove that the circled numbers in the following Pascal-style triangle are u_n. Here each number is the sum of two: one from above and one in the same row in the direction of 0.

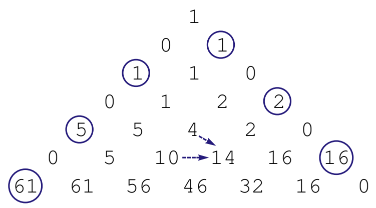

FIGURE 1. Triangle to compute numbers u_n.

3) Let $T_n(x, y)$ be the Tutte polynomial of K_n. Prove that $u_n = |T_{n+1}(1, -1)|$.

4) In a spanning tree $t \in K_n$ we say that vertices i and j form an *inversion* if $i < j$ and j lies on the shortest path from i to 1. Let $\mathrm{inv}(t)$ be the number of inversions in t. Define

$$f_n(q) = \sum_{t \in K_n} q^{\mathrm{inv}(t)}$$

Express $f_n(q)$ via $T_n(1, y)$.

5) Let P_n be a polytope in $\mathbb{R}^d$ defined by inequalities $x_i \geq 0$, $1 \leq i \leq n$, and

$$x_i + x_{i+1} \leq 1, \quad 1 \leq i < n.$$

$a)$ Compute the number of integer points in P_n
(*Hint:* find a classical combinatorial interpretation).

$b)$ Compute the volume of P_n
(*Hint:* find a combinatorial interpretation in terms of u_n.)

$c)$ Give a combinatorial interpretation for the number of integer points in $k \cdot P_n$, generalizing part $b)$. Here $k \cdot X = \{k \cdot x \mid x \in X\}$, and $k \in \mathbb{N}$.

6) Ex. 70 on p. 177 in *Bollobas*, MGT.

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