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Pak

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Linear Algebra Methods in Combinatorics

Thm (Babai - Frankl 1988) "Oddtown Theorem"
$A_1, \dots A_m \subset [n]$, $|A_i| = \text{odd}$ $|A_i \cap A_j| = \text{even}$ $i \neq j$
then $m \leq n$

Pf: $v_1, \dots v_m \in \mathbb{F}_2^n$ $v_i = \chi(A_i)$ $v_i \cdot v_j = \begin{cases} 1 & i = j \\ 0 & o/w \end{cases}$ (in $\mathbb{F}_2$)

Claim: $v_1 \dots v_m$ lin. indep.
pf: $\lambda_1 v_1 + \dots + \lambda_m v_m = 0$ $(\lambda_1 v_1 + \dots + \lambda_m v_m) \cdot v_i = \lambda_i$
$\Rightarrow \lambda_i = 0\ \forall i$ ✓

So $m \leq n$ ✓

Thm (Fisher's inequality) (see Sect. 14 of Jukna)
$A_1, \dots A_m \subseteq [n]$ distinct, $|A_i \cap A_j| = k$ $i<j$
Then $m \leq n$.

Pf: Claim: $v_1 \dots v_m$ $\in \mathbb{R}^n$ lin indep
pf: $\lambda_1 v_1 + \dots + \lambda_m v_m = 0$ Clearly $(v_i, v_i) = |A_i|$,
$\exists$ At most one i s.t. $|A_i| = k$ $(v_i, v_j) = k$ $i<j$
$0 = (\lambda_1 v_1 + \dots + \lambda_m v_m, \lambda_1 v_1 + \dots + \lambda_m v_m) = \sum_{i=1}^m \lambda_i^2 |A_i| + 2 \sum_{i<j} \lambda_i \lambda_j k$
$= \sum_{i=1}^m \lambda_i^2 (|A_i| - k) + k \left(\sum_{i=1}^m \lambda_i\right)^2$ ↯ ✓

Thm # pts in $\mathbb{R}^n$ s.t. pairs ~~are~~ have only two distances
is $\leq \binom{n}{2} + 3n + 2$

Pf: $D_i = (\text{distance } i)^2$ $A = \{a_1, \dots a_m\}$ 2-dist set
Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $f_i(x) = (\|x - a_i\|^2 - D_1)(\|x - a_i\|^2 - D_2)$
Then $f_i(a_j) = 0$ $\forall i \neq j$, $f_i(a_i) = D_1 D_2 \neq 0$
By lemma, f_i lin. indep. (Lemma: $f_1, \dots f_m : X \to \mathbb{R}$
and $f_i(v_i) \neq 0$ $f_i(v_j) = 0$ for
some $v_1, \dots v_m \in X$ $\Rightarrow f_1, \dots f_m$ lin. indep
Pf: BWOC ✓)

Here's a basis for the space spanned by f_i:
$\left(\sum x_i^2\right)^2$, $\left(\sum x_i^2\right) x_j$, $x_i x_j$, x_i^2, x_i, 1
Total size of basis is $\binom{n}{2} + 3n + 2$.