18.319, Fall 2005. Problem Set 1

Instructions: Solve your favourite problems from the list below. Open problems are marked with ✰; hard (but feasible) problems are marked with $\star$.

1. Find all planar point configurations with n points that determine exactly n distinct lines for $n \in \mathbb{N}$, $n \geq 4$.

2. (Motzkin, 1951)

   (a) For every $n \in \mathbb{N}$, $n \geq 6$, find n points in $\mathbb{R}^3$, not all in a plane, such that the plane determined by any three noncollinear points contains at least four points.

   (b) Find a finite point set in the complex plane $\mathbb{C}^2$ that does not lie in a complex line and the complex line determined by any two points contains at least three points. $\star$

3. (Motzkin, 1951) Given a point set P in $\mathbb{R}^d$, $d \geq 2$, we say that a hyperplane h is *ordinary* if all but at most one points of $h \cap P$ lie in a $(d-2)$ dimensional affine subspace. Show that there is an ordinary hyperplane for any finite set of points in $\mathbb{R}^d$, $d \geq 2$.

4. (Kelly-Moser, 1958) Find 7 points in the plane with exactly 3 ordinary lines.

   (McKee, 1968) Find 13 points in the plane with exactly 6 ordinary lines.

5. (Jamison, 1986) Given a set V of n points in the plane, not all on a line, show that there is a connected graph drawn in the plane with straight line edges such that its vertex set is V and its edges have pairwise different slopes.

6. (Jamison, 1987) Given a set V of $n \in \mathbb{N}$ points in the plane, no three of which are collinear, and an (abstract) graph G with n vertices. Is there a straight line embedding of G into the plane such that the vertices of G are mapped onto V and the edges of G are pairwise nonparallel if

   (a) G is a path and V forms a regular n-gon;

   (b) G is a path; $\star$

   (c) G is a tree and V is in convex position; ✰

   (d) G is a tree? ✰

7. (Hopf-Pannwitz, 1934) Show that any n points in the plane, no three of which are collinear, determine at most n pairwise intersecting (closed) line segments.

8. Given a set P of $2n$ noncollinear points in the plane, let $h(P)$ denote the number of its halving lines (i.e., lines spanned by P such that either of their open halfplanes contains less than n points). Let $d_1, d_2, \ldots, d_{h(P)}$ be the number of points on each of these halving lines. Give a lower bound for the number of distinct slopes determined by P in terms of $d_1, d_2, \ldots, d_{h(P)}$.

9. How many distinct slopes are determined by the point set $\{(a,b) \in \mathbb{N}^2 : 1 \leq a, b \leq n\}$ (i.e., the $n \times n$ integer lattice section)? How is it about $\{(a,b,c) \in \mathbb{N}^3 : 1 \leq a, b, c \leq n\}$ in three dimensions?

Brush up exercise on point-line duality in the plane. The point-line duality is a bijection between points and nonvertical lines in the Euclidean plane defined by

$$\begin{aligned} p(a,b) &\longleftrightarrow p^* : y + ax + b = 0, \\ \ell : y + ax + b = 0 &\longleftrightarrow \ell^*(a,b). \end{aligned}$$

- Show that point p is incident to line ℓ if and only if point ℓ^* is incident to line p^*.
- Show that point p lies above line ℓ if and only if line p^* passes below point ℓ^*.
- What is the dual of a line segment p_1p_2?
- Formulate the dual statement for "The closed line segments p_1p_2 and q_1q_2 intersect."
- Formulate the dual statement for "Point sets A and B are separated by a vertical line."
- What is the dual of the inner diagonals of two point sets, A and B, separated by a vertical line?