18.319, Fall 2005. Problem Set 3

Instructions: Solve your favourite problems from the list below. Open problems are marked with ✰; hard (but feasible) problems are marked with $\star$.

1. (Székely, 1997) For a simple graph G and $k \in \mathbb{N}$, let kG denote the graph obtained from G by replacing every edge by k parallel edges. Show that $k^2 \cdot \mathrm{cr}(G) = \mathrm{cr}(kG)$.

2. For $t \in \mathbb{N}$, we are given an alphabet of size t and a string s of 2^t letters. Find a (nonempty) substring s' of consecutive letters from s such that each letter occurs in s' an even number of times.

3. (Djidjev and Vrt́o, 2003) Given a simple topological graph $G(V, E, D)$, where D stands for the planar embedding of the graph, let $\ell(D)$ be the *maximum* number of edges crossed by a vertical line. The *cut width* $\mathrm{cw}(G)$ of a simple graph $G(E, V)$ is defined as the *minimum* $\ell(D)$ over all drawings D in which the vertices have distinct x-coordinates. (a) Show that $\mathrm{bw}(G) \leq \mathrm{cw}(G)$. (b) Show that
$$\mathrm{cr}(G) = \Omega(\mathrm{cw}^2(G)) - O\left(\sum_{p \in V} \deg^2(p)\right).$$

4. (a) For every $k \in \mathbb{N}$, construct a graph whose crossing number is k.

   (b) For every $k \in \mathbb{N}$, find a graph $G(V, E)$ and an edge $pq \in E$ such that $\mathrm{cr}(G) = k$ but $G'(V, E \setminus \{pq\})$ is planar.

   (c) Find 3-regular graphs $G(E, V)$ such that $\mathrm{cr}(G) = 1$, but $G'(V, E \setminus \{pq\})$ is planar for any edge $pq \in E$.

   (d) Every 3-regular graph $G(V, E)$ has an edge $pq \in E$ such that the crossing number of $G'(V, E \setminus \{pq\})$ is at least $\Omega(\mathrm{cr}(G)) - O(1)$. $\star$

   (e) (Richter and Thomassen, 1993) Show (d) for simple graphs. $\star$

5. K is a complete geometric graph with n vertices, each edge is colored red or blue.

   (a) (Bialostocki and Dierker) Show that K contains a monochromatic spanning tree with pairwise non-crossing edges if V forms the vertex set of a convex n-gon.

   (b) (Károlyi et al., 1997) Show that K contains a monochromatic spanning tree with pairwise non-crossing edges. $\star$

   (c) Show that K contains $\lfloor (n+1)/3 \rfloor$ pairwise disjoint edges of the same color.

   (d) Color the edges of a complete graph K_n with two colors such that there are no $\lfloor (n+1)/3 \rfloor + 1$ pairwise disjoint edges of the same color.

6. Let lin-cr(G) denote the *rectilinear crossing number* of G, which is the minimum number of crossings in a drawing of G with all edges drawn as straight line segments.

   (a) Show that if $\mathrm{cr}(G) = 1$, then lin-cr$(G) = 1$.

   (b) Find a simple graph where $\mathrm{cr}(G) <$ lin-cr(G). $\star$