18.319, Fall 2005. Problem Set 5

Open problems are marked with ✰; hard (but feasible) problems are marked with $\star$. Problems from Matoušek's textbook on Discrete Geometry are marked with 🚲.

1. (Asano et al., 1999) You are given a finite set S of pairwise disjoint line segments in the plane and a curve γ that intersects every segment in S. Consider a point p outside of the convex hull of γ. Show that there is a point $q \in \gamma$ such that the line segment pq is disjoint from all segments of S.

2. You are given a set P of n noncollinear points in the plane.
   (a) (Ungar) Show that there is a line ℓ such that P determines at least one but at most $(n-1)/2$ line segments parallel to ℓ.
   (b) (Burton & Purdy) Show that there are two points $p, q \in P$ such that the number of distinct distances of the points of P from the line pq is at least $.4n - O(1)$.

3. Suppose that for every $n \in \mathbb{N}$, n even, you can find an n-element point set in the plane with $f(n)$ halving edges. (a) Show that for every $k, n \in \mathbb{N}$, $k \leq n/2$, there are n points in the plane such that the number of k-edges is at least $\Omega(\lfloor n/2k \rfloor f(2k))$.
   (b) Show that for infinitely many $n \in \mathbb{N}$, there are n points in $\mathbb{R}^3$ with $\Omega(nf(n))$ halving triangles. 🚲

4. (Lovász) You are given a set P of $2n$ points in the plane in general position, and a vertical line ℓ having exactly k points on the left and exactly $2n-k$ points on the right. Prove that ℓ crosses exactly $\min(k, 2n-k)$ halving edges of P.

5. (Welzl) Let $t(n)$ denote the maximum number of halving triangles for a set of n points in $\mathbb{R}^3$. Show that for every $n \in \mathbb{N}$, there is a set P_n of n points in convex position in $\mathbb{R}^3$ that has $t(n)$ halving triangles. $\star$

6. (Pach & Pinchasi) R is a set of n red points, and B is a set of n blue points in the plane such that $B \cap R = \emptyset$ and $R \cup B$ is in general position. A line ℓ is called *balanced* if ℓ passes through a red point and a blue point, and each open half-plane bounded by ℓ contains the same number of blue points as red points. Show that there are at least n balanced lines.

7. Consider an arrangement of n circles in the plane and a parameter $r \in \mathbb{N}$, $1 \leq r \leq n/2$. Show that there is a partition of the plane into $O(r^2 \log^2 r)$ regions, each bounded by a finite number of straight line segments and circular arcs, such that the interior of each region intersects at most n/r circles. 🚲

8. (a) Determine the VC-dimension of the range space $(\mathbb{R}^2, T)$, where T is the set of all triangles in the plane. 🚲
   (b) (Kalai & Matoušek) Let S be a simply connected compact set in the plane. For every point $s \in S$, let $V(s) = \{p \in S : \text{the segment } ps \text{ lies in } S\}$ be the visibility range of s. Show that the range space $(S, \{V(s) : s \in S\})$ has finite VC-dimension.