Course 18.327 and 1.130
Wavelets and Filter Banks

Refinement Equation: Iterative and Recursive Solution Techniques; Infinite Product Formula; Filter Bank Approach for Computing Scaling Functions and Wavelets

Solution of the Refinement Equation

$$\phi(t) = 2\sum_{k=0}^{N} h_0[k]\,\phi(2t-k)$$

First, note that the solution to this equation may not always exist! The existence of the solution will depend on the discrete-time filter $h_0[k]$.

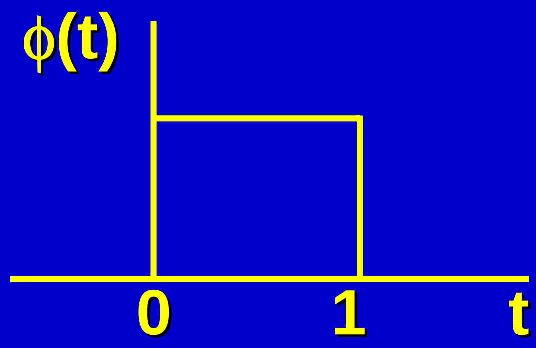
If the solution does exist, it is unlikely that $\phi(t)$ will have a closed form solution. The solution is also unlikely to be smooth. We will see, however, that if $h_0[n]$ is FIR with

$$h_0[n] = 0 \quad \text{outside} \quad 0 \leq n \leq N$$

then $\phi(t)$ has compact support:

$$\phi(t) = 0 \quad \text{outside} \quad 0 < t < N$$

Approach 1 **Iterate the box function**



$\phi^{(0)}(t) \;=\;$ box function on [0 , 1]

$$\phi^{(i + l)}(t) = 2\sum_{k=0}^{N} h_0[k]\, \phi^{(i)}(2t - k)$$

If the iteration converges, the solution will be given by

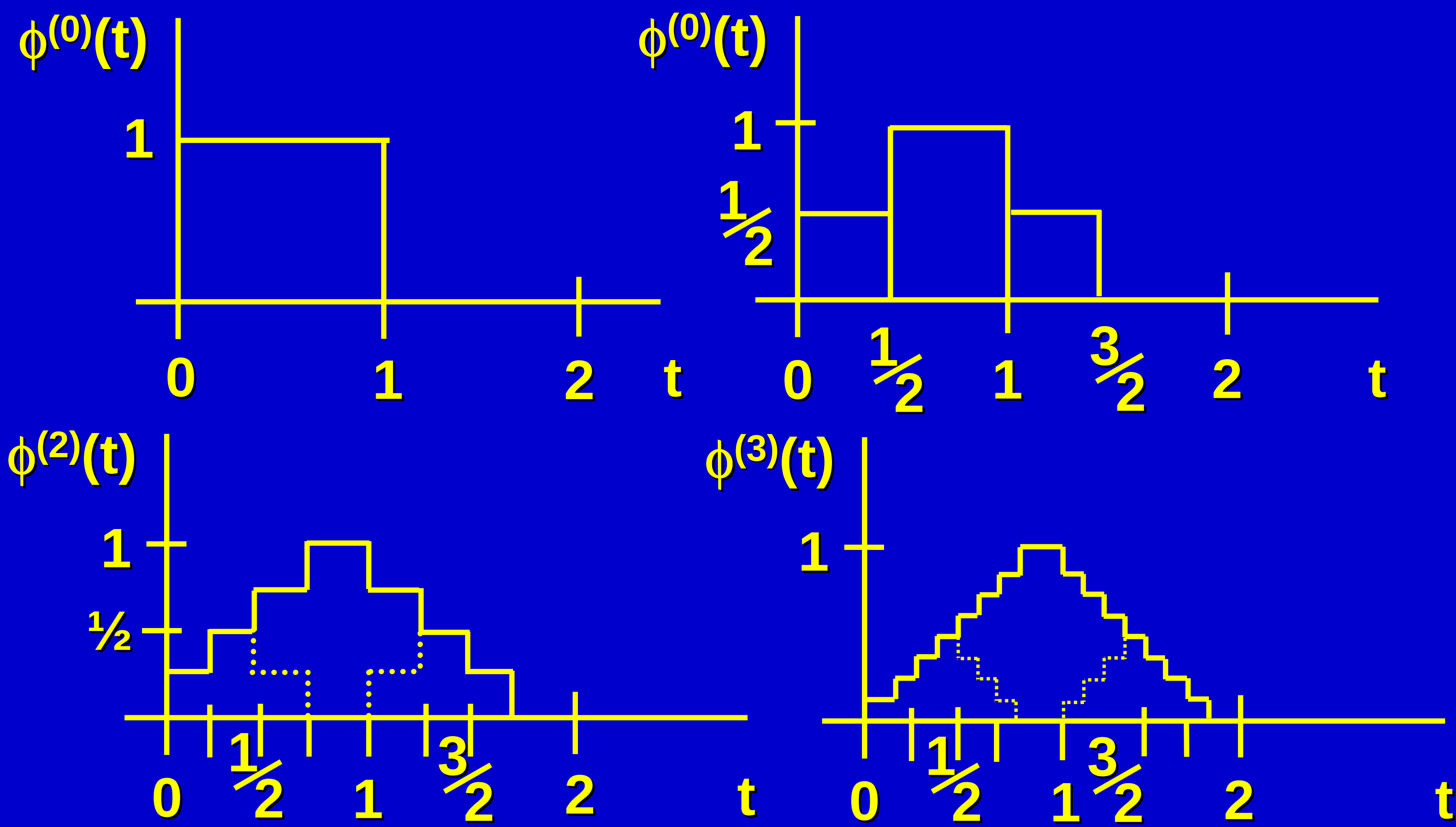
$$\lim_{i \to \infty} \phi^{(i)}(t)$$

This is known as the **cascade algorithm**.

Example: suppose $h_0[k] = \{¼, ½, ¼\}$

$$\phi^{(i+1)}(t) = ½\,\phi^{(i)}(2t) + \phi^{(i)}(2t-1) + ½\,\phi^{(i)}(2t-2)$$

Then



Converges to the hat function on [0, 2]

Approach 2 Use recursion

First solve for the values of $\phi(t)$ at integer values of t.

Then solve for $\phi(t)$ at half integer values, then at quarter integer values and so on.

This gives us a set of discrete values of the scaling function at all dyadic points $t = n/2^i$.

At integer points:

$$\phi(n) = 2 \sum_{k=0}^{N} h_0[k]\, \phi(2n - k)$$

Suppose N = 3

$$\phi(0) = 2 \sum_{k=0}^{3} h_0[k]\, \phi(-k)$$

$$\phi(1) = 2 \sum_{k=0}^{3} h_0[k]\, \phi(2-k)$$

$$\phi(2) = 2 \sum_{k=0}^{3} h_0[k]\, \phi(4-k)$$

$$\phi(3) = 2 \sum_{k=0}^{3} h_0[k]\, \phi(6-k)$$

Using the fact that $\phi(n) = 0$ for $n < 0$ and $n > N$, we can write this in matrix form as

$$\begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix} = 2 \begin{bmatrix} h_0[0] & & & \\ h_0[2] & h_0[1] & h_0[0] & \\ & h_0[3] & h_0[2] & h_0[1] \\ & & & h_0[3] \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix}$$

Notice that this is an eigenvalue problem

$$\lambda\Phi = A\Phi$$

where the eigenvector is the vector of scaling function values at integer points and the eigenvalue is $\lambda = 1$.

Note about normalization:
Since $(A - \lambda I)\,\Phi = 0$ has a non-unique solution, we must choose an appropriate normalization for Φ
The correct normalization is

$$\sum_n \phi(n) = 1$$

This comes from the fact that we need to satisfy the partition of unity condition, $\sum_n \phi(x-n) = 1$.

At half integer points:

$$\phi\,(n/2) = 2 \sum_{k=0}^{N} h_0[k]\, \phi\,(n-k)$$

So, for N = 3, we have

$$\begin{bmatrix} \phi(1/2) \\ \phi(3/2) \\ \phi(5/2) \end{bmatrix} = 2 \begin{bmatrix} h_0[1] & h_0[0] & & \\ h_0[3] & h_0[2] & h_0[1] & h_0[0] \\ & & h_0[3] & h_0[2] \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix}$$

Scaling Relation and Wavelet Equation in Frequency Domain

$$\phi(t) = 2 \sum_k h_0[k]\, \phi(2t - k)$$

$$\int_{-\infty}^{\infty} \phi(t) e^{-i\Omega t}\, dt = 2 \sum_k h_0[k] \int_{-\infty}^{\infty} \phi(2t - k)\, e^{-i\Omega t}\, dt$$

$$= 2 \sum_k h_0[k]\, \tfrac{1}{2} \int_{-\infty}^{\infty} \phi(\tau) e^{-i\Omega(\tau + k)/2}\, d\tau$$

$$= \sum_k h_0[k] e^{-i\Omega k/2} \int_{-\infty}^{\infty} \phi(\tau)\, e^{-i\Omega\tau/2}\, d\tau$$

$$\text{i.e. } \hat{\phi}(\Omega) = H_0\left(\frac{\Omega}{2}\right) \cdot \hat{\phi}\left(\frac{\Omega}{2}\right)$$

$$= H_0\left(\frac{\Omega}{2}\right) \cdot H_0\left(\frac{\Omega}{4}\right) \cdot \hat{\phi}\left(\frac{\Omega}{4}\right)$$

$$\vdots$$

$$= \left\{ \prod_{j=1}^{\infty} H_0\left(\frac{\Omega}{2^j}\right) \right\} \hat{\phi}(0)$$

$$\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(t)\, dt = 1 \text{ (Area is normalized to 1)}$$

So

$$\hat{\phi}(\Omega) = \prod_{j=1}^{\infty} H_0\left(\frac{\Omega}{2^j}\right)$$ Infinite Product Formula

Similarly

$$w(t) = 2 \sum_k h_1[k]\, \phi(2t - k)$$

leads to

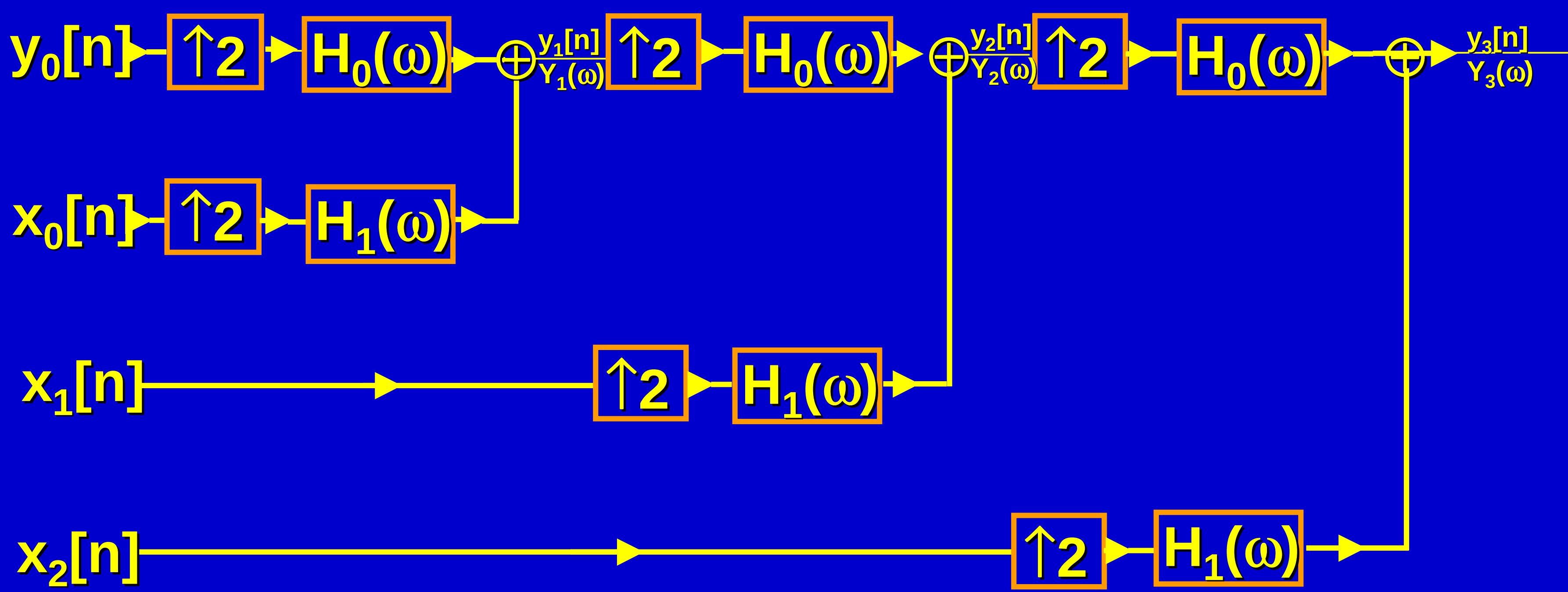
$$\hat{w}(\Omega) = H_1\left(\frac{\Omega}{2}\right) \hat{\phi}\left(\frac{\Omega}{2}\right)$$

Desirable properties for $H_0(\omega)$:

- $H(0) = 1$, so that $\hat{\phi}(0) = 1$
- $H(\omega)$ should decay to zero as $\omega \rightarrow \pi$,

so that $\int_{-\infty}^{\infty} |\hat{\phi}(\Omega)|^2 d\,\Omega < \infty$

Computation of the Scaling Function and Wavelet – Filter Bank Approach



Normalize so that $\sum_{n} h_0[n] = 1.$

i. Suppose $y_0[n] = \delta[n]$ and $x_k[n] = 0$.

$$Y_0(\omega) = 1$$
$$Y_1(\omega) = Y_0(2\omega)\, H_0(\omega) = H_0(\omega)$$
$$Y_2(\omega) = Y_1(2\omega)\, H_0(\omega) = H_0(2\omega) H_0(\omega)$$
$$Y_3(\omega) = Y_2(2\omega)\, H_0(\omega) = H_0(4\omega)\, H_0(2\omega)\, H_0(\omega)$$

After K iterations:

$$Y_K(\omega) = \prod_{k=0}^{K-1} H_0(2^k \omega)$$

What happens to the sampling period?

Sampling period at input = $T_0 = 1$ (say)

Sampling period at output = $T_K = ½^K$

Treat the output as samples of a continuous time signal, $y_K^c(t)$, with sampling period $\frac{1}{2^K}$:

$$y_K[n] = \frac{1}{2^K} y_K^c(n/2^K)$$

$$\Rightarrow Y_K(\omega) = \hat{Y}_K^c(2^K\omega) \quad ; \quad -\pi \le \omega \le \pi$$

($y_K^c(t)$ is chosen to be bandlimited)

Replace $2^K\omega$ with Ω:

$$\hat{Y}_K^c(\Omega) = Y_K(\Omega/2^k) = \prod_{k=0}^{K-1} H_0(\Omega/2^{K-k}) = \prod_{j=1}^{K} H_0(\Omega/2^j) \quad ;$$

$$-2^K\pi \le \Omega \le 2^K\pi$$

So

$$\lim_{k\to\infty} \hat{Y}_K^c(\Omega) = \prod_{j=1}^{\infty} H_0(\Omega/2^j) = \hat{\phi}(\Omega)$$

$\Rightarrow$ $2^K y_K[n]$ converges to the samples of the scaling function, $\phi(t)$, taken at $t = n/2^K$.

ii. Suppose $y_0[n] = 0$, $x_0[n] = \delta[n]$ and all other $x_k[n] = 0$

$$Y_K(\omega) = H_1(2^{K-1}\omega) \prod_{k=0}^{K-2} H_0(2^k\omega)$$

Then

$$\hat{Y}_K^c(\Omega) = Y_K(\Omega/2^K) = H_1\left(\frac{\Omega}{2}\right) \prod_{k=0}^{K-2} H_0(\Omega/2^{K-k})$$

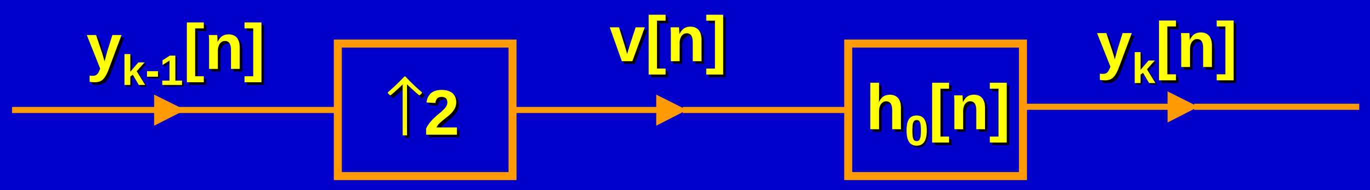
$$= H_1\left(\frac{\Omega}{2}\right) \prod_{j=1}^{K-1} H_0\left(\frac{1}{2}\cdot\frac{\Omega}{2^j}\right)$$

So

$$\lim_{K\to\infty} \hat{Y}_K^c(\Omega) = H_1(\Omega/2)\,\hat{\phi}(\Omega/2) = \hat{w}(\Omega)$$

$\Rightarrow$ $2^K y_K[n]$ converges to the samples of the wavelet, $w(t)$, taken at $t = n/2^K$.

Support of the Scaling Function



length {v[n]} = 2 • length {y_{k-1}[n]} - 1

Suppose that

$h_0[n] = 0$ for $n < 0$ and $n > N$

$\Rightarrow$length {y_k[n]} = length {v[n]} + length {h_0[n]} – 1

= 2 • length {y_{K-1}[n]} + N – 1

Solve the recursion with length {y_0[n]} = 1

So

length {y_k[n]} = $(2^K - 1)N + 1$

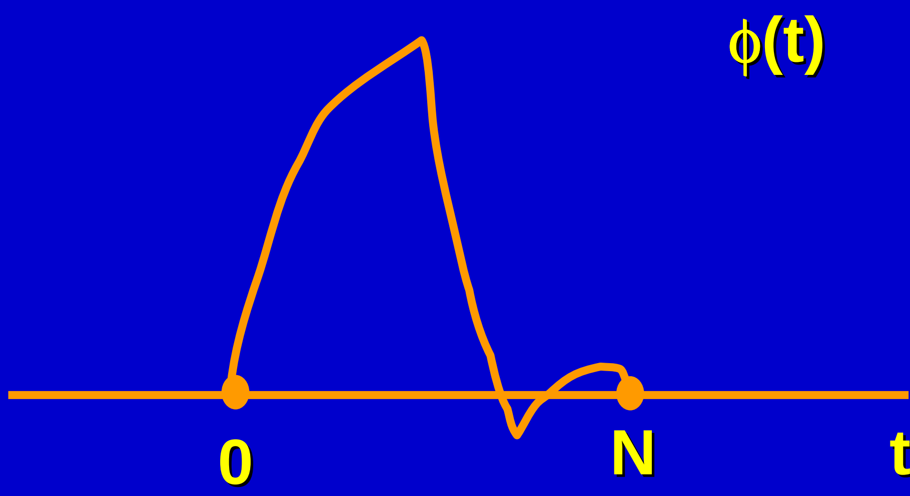
i.e. length $\{y_K^c(t)\} = T_K \cdot \text{length}\ \{y_K[n]\}$

$$= \frac{(2^K - 1)N + 1}{2^K}$$

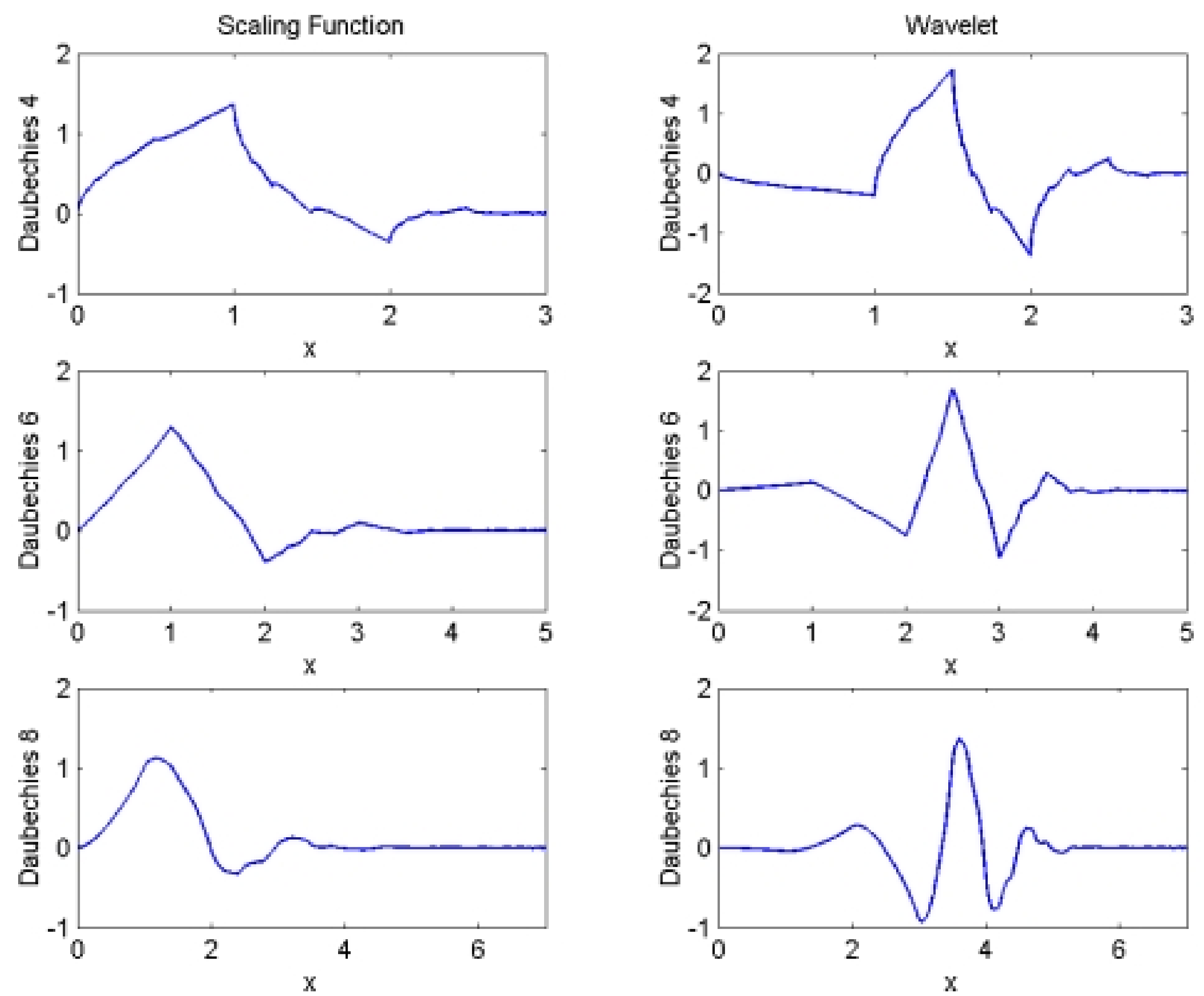
$$= N - \frac{N-1}{2^K}$$

$\lim\ K \to \infty$

length $\{\phi(t)\} = N$



So the scaling function is supported on the interval [0, N]

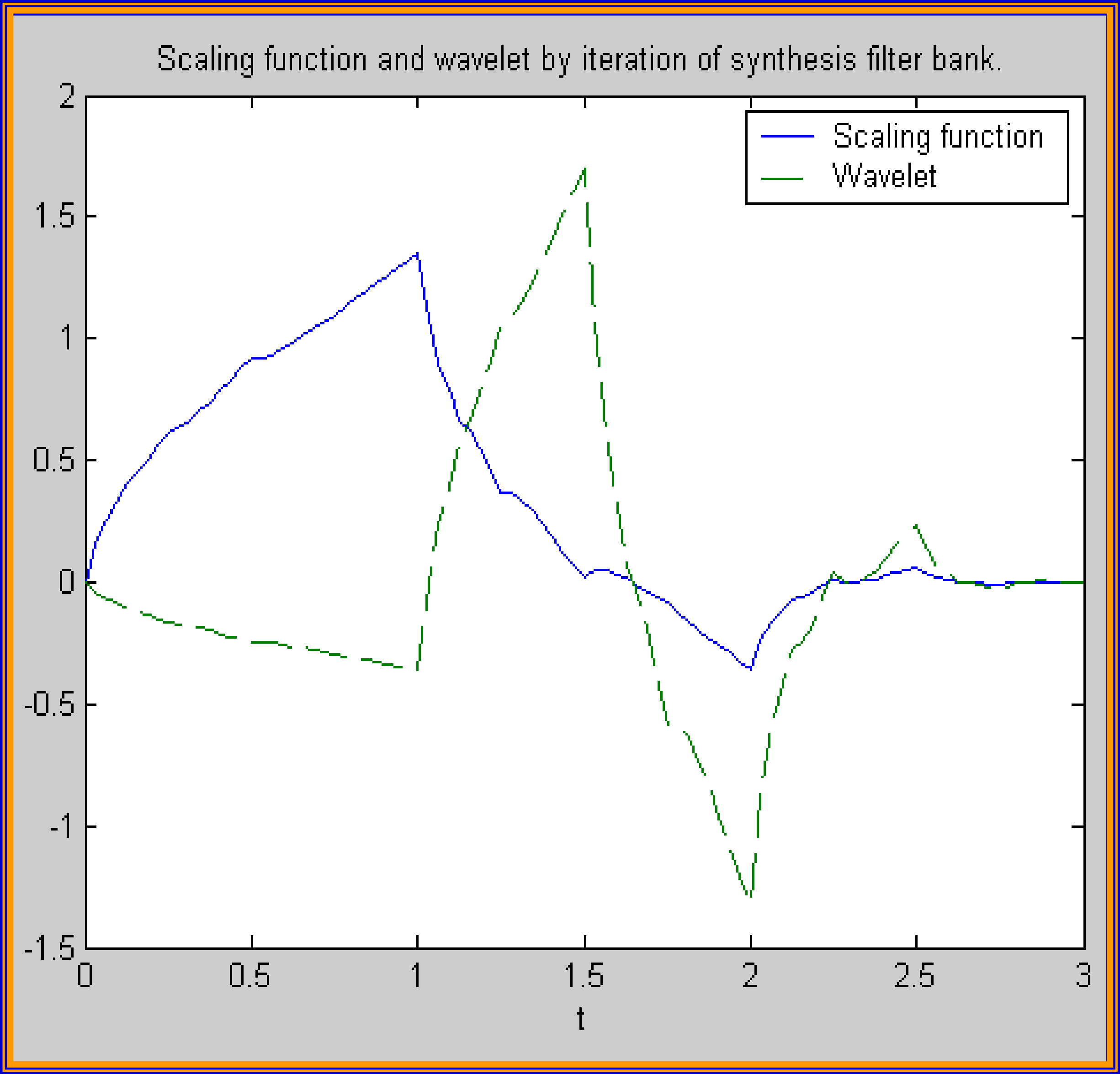

Scaling Function
Wavelet
Daubechies 4
Daubechies 6
Daubechies 8
x

Matlab Example 6

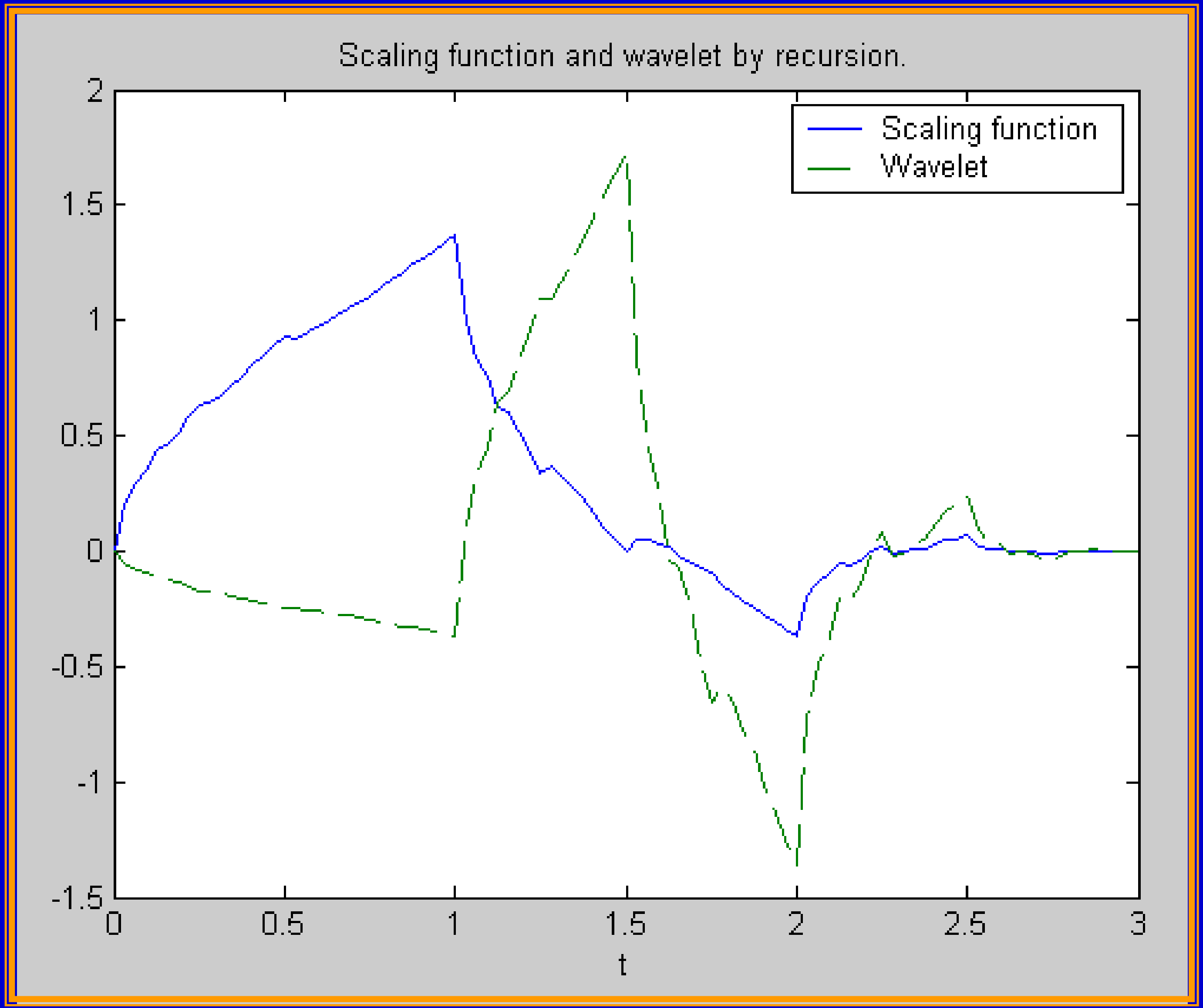
Generation of orthogonal scaling functions and wavelets

MATLAB M-file MATLAB M-file

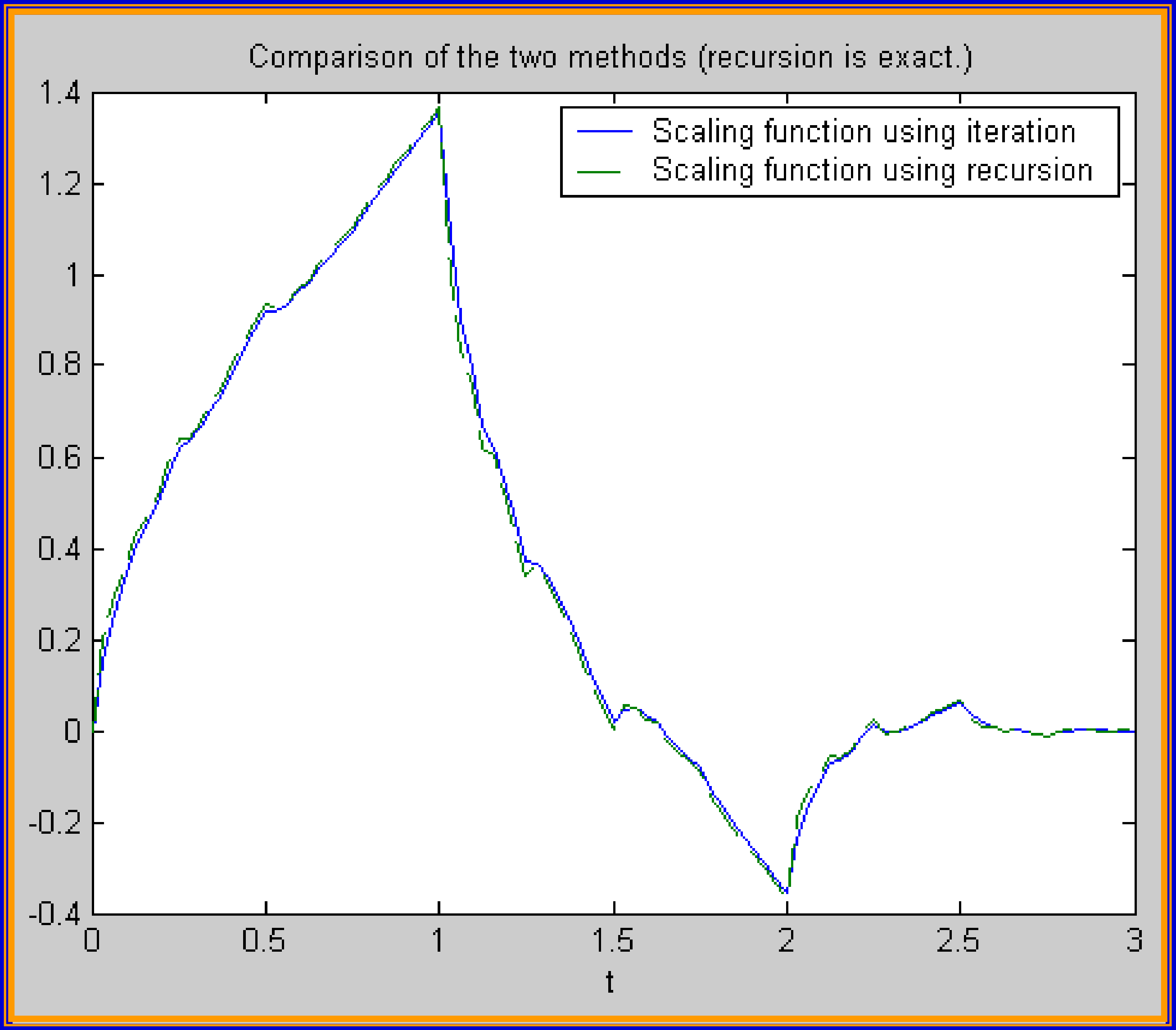
By Inverse DWT



By Recursion



Comparison



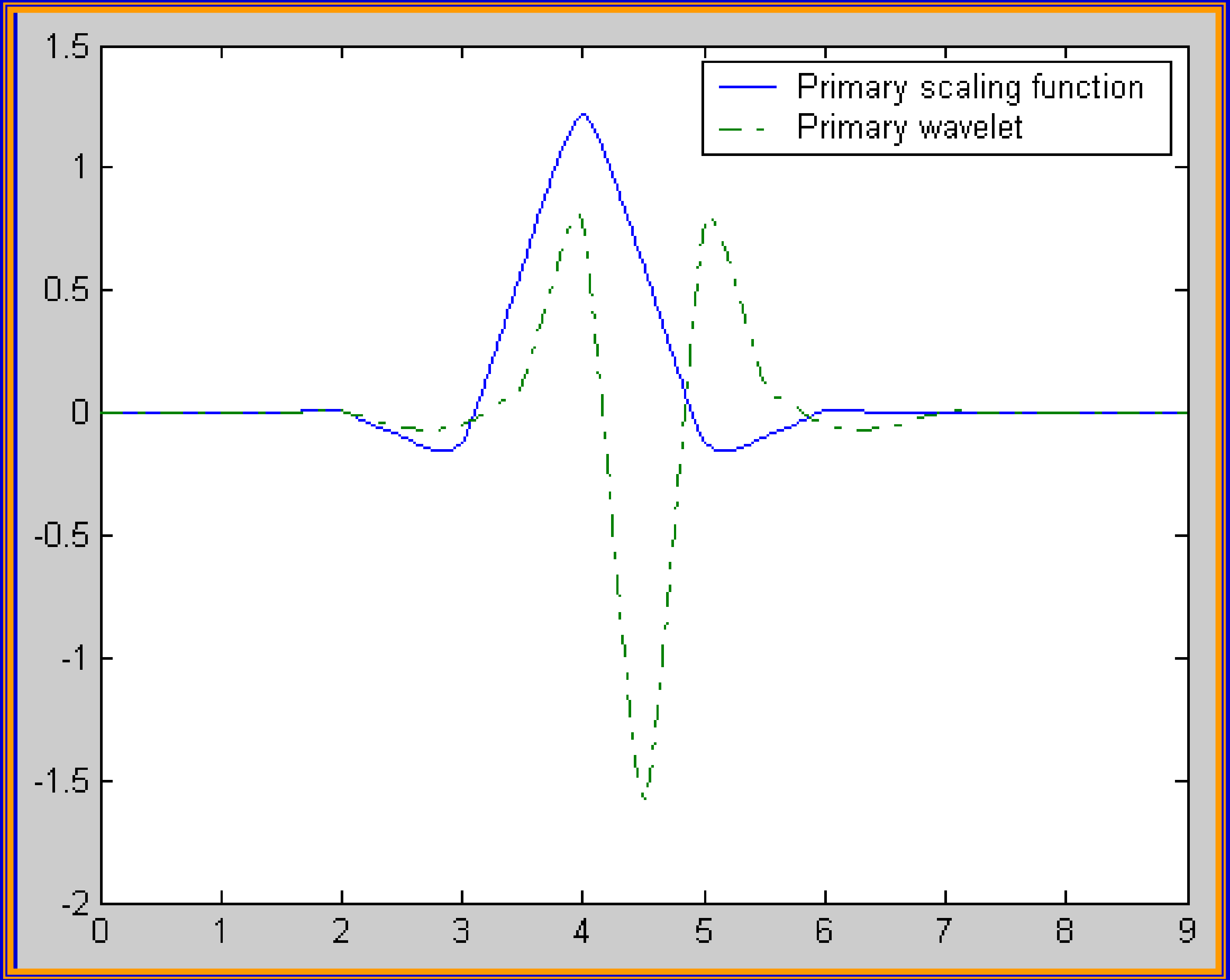
Comparison of the two methods (recursion is exact.)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
0.5
1
1.5
2
2.5
3
t
Scaling function using iteration
Scaling function using recursion

Matlab Example 7

Generation of biorthogonal scaling functions and wavelets.

MATLAB M-file

Primary Daub 9/7 Pair



Dual Daub 9/7 Pair