CONDITION NUMBER

re $\underline{A}\vec{x} = \vec{b}$

Here take $\underline{A}$ to be a real, symmetric, 5 × 5 matrix.

1. Random matrix. When all 15 independent elements of $\underline{A}$ are selected randomly from a Gaussian distribution with mean $\overline{\langle a_{ij} \rangle} = 0$ and dispersion $\langle a_{ij}^2 \rangle = 1$, four separate numerical experiments involving 50 such matrices apiece indicate that the calculated

$$CN \equiv \max|\lambda|/\min|\lambda|$$

seem to be distributed about so:

Exp.	#1	#2	#3	#4
largest	424	111	146	779
90% ile	70	37	23	30
75% ile	28	18	15	17
median	**11**	**9**	**8**	**8**
25% ile	6	6	5	4
10% ile	4	4	3	3
smallest	2.2	2.8	2.5	2.9

2. Loaded string. On that basis, the well-known matrix on the right, with eigenvalues 1, 2, 3, and $2 \pm \sqrt{3}$ — and therefore

$$\boxed{CN = 13.93}$$

— seems only mildly perverse.

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

3. Loaded beam. Considerably more irritating, owing to such likely loss of accuracy upon inversion, is the matrix:

$$\begin{bmatrix} 6 & -4 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 1 & -4 & 6 \end{bmatrix}$$

Its eigenvalues are 0.21207, 1.4689, 4.6790, 9.5311 and 14.109; its

$$\boxed{CN = 66.53}.$$

4. Loaded dice! Quite appalling in this sense is the Hilbert matrix, whose (i,j)-th element is defined as the reciprocal of the sum i+j-1 . Its eigenvalues are

1.56705, 2.08534E-1, 1.14075E-2, 3.05898E-4, 3.28793E-6

and its condition number is

CN = 476607 (!).

Note: If we had instead been dealing with 10×10 matrices, the various CN's would have emerged as

R: 19 (median) S: 48.37 B: 633.3 and H: a modest 1.6×10^{13}.

What does all this portend in practice?

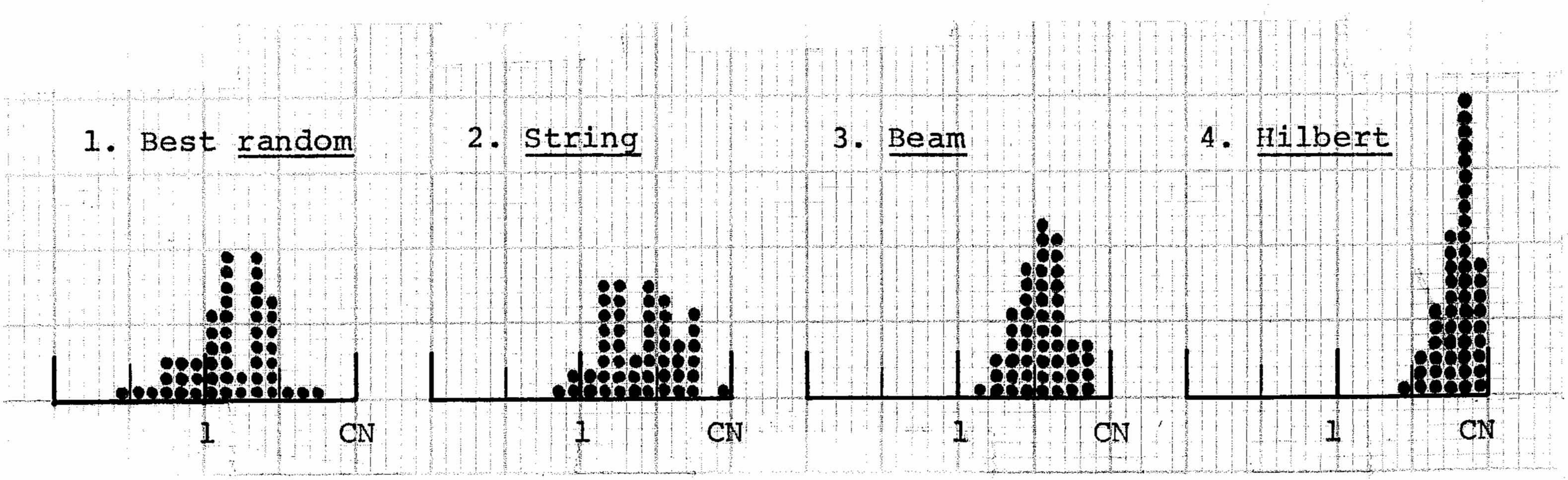
Obviously the "typical" relative error in the inferred $\vec{x}$ is not going to be quite as bad as CN times the relative error in the given vector, or $\delta\vec{b}$ — after all, the CN was invented by pessimists! But just how are the actual ratios

$$\mu = \frac{|\delta\vec{x}|\ |\vec{b}|}{|\vec{x}|\ |\delta\vec{b}|}$$

usually distributed within their conceivable range $\frac{1}{CN} \le \mu \le CN$?

For this purpose, let us simply divide each such range into 20 logarithmically equal intervals or bins — e.g., ... 1/2 to 1, 1 to 2, 2 to 4, ... if CN=1024. Let us also produce, in their respective vector spaces, completely isotropic and random $\vec{x}$ and $\delta\vec{b}$.

I have now conducted 50 such numerical experiments apiece for each of the four matrices above. (Here "random" was defined as that unusually lucky matrix whose CN = only 2.2.) The resulting histograms pretty much tell their own story: awful remains awful.



Median μ: 1.13 3.0 10 40000