Basic Idea: For any polynomial $P_n(x)$ with real coefs., look for roots two at a time via quadratic factors … so that

$$P_n(x) \equiv a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$
$$= (x^2 - rx - s)(b_0 x^{n-2} + b_1 x^{n-3} + \dots) .$$

Algebra: Of course,

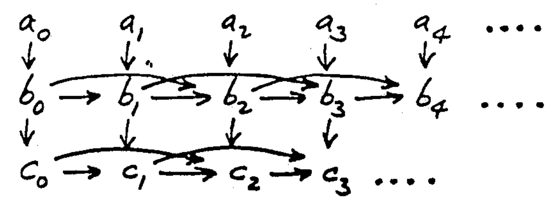
$$\begin{aligned} b_0 &= a_0 \\ b_1 &= a_1 + r b_0 \\ b_2 &= a_2 + r b_1 + s b_0 \\ b_3 &= a_3 + r b_2 + s b_1 \\ b_4 &= a_4 + r b_3 + s b_2 \end{aligned} \quad \text{etc.}$$

and we wish to find r and s such that at once $b_{n-1} = 0$ and $b_n = 0$. But notice also that

$$\frac{\partial b_0}{\partial r} = 0 \quad \text{NIL}$$
$$\frac{\partial b_1}{\partial r} = b_0 + r\frac{\partial b_0}{\partial r}$$
$$\frac{\partial b_2}{\partial r} = b_1 + r\frac{\partial b_1}{\partial r} + s\frac{\partial b_0}{\partial r}$$
$$\frac{\partial b_3}{\partial r} = b_2 + r\frac{\partial b_2}{\partial r} + s\frac{\partial b_1}{\partial r}$$
etc.

$$\frac{\partial b_1}{\partial s} = \frac{\partial b_0}{\partial s} = 0$$
$$\frac{\partial b_2}{\partial s} = b_0 + r\frac{\partial b_1}{\partial s} + s\frac{\partial b_0}{\partial s}$$
$$\frac{\partial b_3}{\partial s} = b_1 + r\frac{\partial b_2}{\partial s} + s\frac{\partial b_1}{\partial s}$$
$$\frac{\partial b_4}{\partial s} = b_2 + r\frac{\partial b_3}{\partial s} + s\frac{\partial b_2}{\partial s}$$
etc.

Bairstow tableau: Evidently then we can sequentially manufacture all this vital information via



where $c_0 \equiv \frac{\partial b_1}{\partial r} = \frac{\partial b_2}{\partial s}$; $c_1 \equiv \frac{\partial b_2}{\partial r} = \frac{\partial b_3}{\partial s}$; etc.

Example: Again consider $\boxed{P_4(x) = x^4 - 10x^3 + 35x^2 - 50x + 24}$ and likewise again consider the factor (?) $\boxed{x^2 - 2x - 3}$. The resulting tableau then is: I.e.: $r=2$, $s=3$

a:	1	-10	35	-50	24
b:	1	-8	22	(-30)	(30)
c:	1	(-6)	(13)	(-22)	

and $\frac{\partial b_4}{\partial r} = c_3 = -22$

And now notice that $b_3 = -30$, $b_4 = +30$ here.

Also that $\frac{\partial b_2}{\partial s} = c_1 = -6$; $\frac{\partial b_3}{\partial r} = \frac{\partial b_4}{\partial s} = c_2 = 13$

Not too bad, is it?

A.T.

Program PWILK6

```
c     Gives roots of 6-th degree "Wilkinson"  P(x) = (x-1)(x-2)...(x-6) ,
c     via successive quadratic Bairstow factors of type  (x^2 - rx - s) ,
c     starting from the deliberately crummy initial guess  r=3, s=2 .


      implicit double precision (a-h,o-z)

      dimension a(9), b(9), c(9)

      data a / 0,0,1,-21,175,-735,1624,-1764,720 /
      data b / 9*0 /
      data c / 9*0 /

      r = 3.0d0
      s = 2.0d0


      do 49 N=6,2,-2

         write (*,13) N, (a(k), k=3,N+3)
13       format (// ' N =', i2, 5x, 'coefs =', 7f8.1)
         write (*,*)

         do 29 iter=1,99

            write (*,15) iter, r, s
15          format (5x, 'iter =', i3, 5x, 'r, s =', 2f18.12)

               do 19 k=3,N+3
               b(k) = a(k) + r*b(k-1) + s*b(k-2)
               c(k) = b(k) + r*c(k-1) + s*c(k-2)
19             continue

            denom = c(N+1)*c(N+1) - c(N)*c(N+2)
            dr = (c(N) * b(N+3) - c(N+1)*b(N+2)) / denom
            ds = (c(N+2)*b(N+2) - c(N+1)*b(N+3)) / denom
            r = r + dr
            s = s + ds

            if (abs(dr).lt.1.0d-12.and.abs(ds).lt.1.0d-12) go to 31

29       continue

31          do 39 k=1,N+1
            a(k) = b(k)
39          continue

49    continue


      end
```

Results from the given program:

```
N = 6     coefs =    1.0   -21.0   175.0  -735.0  1624.0 -1764.0   720.0

     iter =  1     r, s =    3.000000000000    2.000000000000
     iter =  2     r, s =    7.063897763578    4.249201277955
     iter =  3     r, s =    7.041751841148    -.109268869153
     iter =  4     r, s =    7.026514028303   -2.925256150611
     iter =  5     r, s =    7.015600777514   -4.658564051120
     iter =  6     r, s =    7.007389789965   -5.593990466494
     iter =  7     r, s =    7.001864579791   -5.945367427954
     iter =  8     r, s =    7.000089771537   -5.998802266628
     iter =  9     r, s =    7.000000112944   -5.999999414752
     iter = 10     r, s =    7.000000000000   -6.000000000000


N = 4     coefs =    1.0   -14.0    71.0  -154.0   120.0

     iter =  1     r, s =    7.000000000000   -6.000000000000
     iter =  2     r, s =    7.000000000000   -8.400000000000
     iter =  3     r, s =    7.000000000000   -9.507692307692
     iter =  4     r, s =    7.000000000000   -9.918794607455
     iter =  5     r, s =    7.000000000000   -9.996950479611
     iter =  6     r, s =    7.000000000000   -9.999995364349
     iter =  7     r, s =    7.000000000000   -9.999999999989
     iter =  8     r, s =    7.000000000000  -10.000000000000


N = 2     coefs =    1.0    -7.0    12.0

     iter =  1     r, s =    7.000000000000  -10.000000000000
     iter =  2     r, s =    7.000000000000  -12.000000000000
```

Results from start instead at r=3.1, s=-2.1:

```
N = 6     coefs =    1.0   -21.0   175.0  -735.0  1624.0 -1764.0   720.0

     iter =  1     r, s =    3.100000000000   -2.100000000000
     iter =  2     r, s =    2.983004300197   -1.980963049042
     iter =  3     r, s =    2.999734826043   -1.999620132474
     iter =  4     r, s =    3.000000006672   -1.999999937587
     iter =  5     r, s =    3.000000000000   -2.000000000000


N = 4     coefs =    1.0   -18.0   119.0  -342.0   360.0

     iter =  1     r, s =    3.000000000000   -2.000000000000
     iter =  2     r, s =    5.722955145119   -2.284960422164
     iter =  3     r, s =   11.692665787145   -3.556321606269
     iter =  4     r, s =    9.969109468155   -2.383382721827
     iter =  5     r, s =    9.461467149539   -9.819682145004
     iter =  6     r, s =    9.224514700172  -14.172611695105
     iter =  7     r, s =    9.107984074455  -16.441923974832
     iter =  8     r, s =    9.047948607802  -17.519511981082
     iter =  9     r, s =    9.015601179158  -17.924687600759
     iter = 10     r, s =    9.001852936961  -17.998672506178
     iter = 11     r, s =    9.000017646613  -18.000025958122
     iter = 12     r, s =    9.000000000943  -18.000000002154
     iter = 13     r, s =    9.000000000000  -18.000000000000


N = 2     coefs =    1.0    -9.0    20.0

     iter =  1     r, s =    9.000000000000  -18.000000000000
     iter =  2     r, s =    9.000000000000  -20.000000000000
```