Von Neumann Stability Analysis

Lax-equivalence theorem (linear PDE):

Consistency and stability $\Longleftrightarrow$ convergence

(Taylor expansion) $\uparrow$ consistency; (property of numerical scheme) $\uparrow$ stability

Idea in von Neumann stability analysis:
Study growth of waves e^{ikx}.
(Similar to Fourier methods)

Ex.: Heat equation

$u_t = D \cdot u_{xx}$

Solution:

$u(x,t) = \underbrace{e^{-Dk^2 t}}_{=G(k) \text{ growth factor}} \cdot e^{ikx}$

no growth if $|G(k)| \leq 1 \ \forall k$

FD Scheme:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = D \cdot \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

(Explicit Euler) (Central)

$$\Rightarrow U_j^{n+1} = U_j^n + r \cdot \left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right) \ , \ r = \frac{D\Delta t}{(\Delta x)^2}$$

Insert $u(x, t_n) = e^{ikx}$ into FD scheme:

$$U_j^{n+1} = e^{ik\Delta x \cdot j} + r\left(e^{ik\Delta x\cdot(j+1)} - 2e^{ik\Delta x \cdot j} + e^{ik\Delta x \cdot (j-1)}\right)$$
$$= (1 + r(e^{ik\Delta x} + e^{-ik\Delta x} - 2))e^{ik\Delta x \cdot j}$$
$$= G(k) \cdot e^{ik\Delta x \cdot j}$$

Growth factor: $G(k) = 1 - 2r \cdot (1 - \cos(k\Delta x))$
FD scheme stable, if $|G(k)| \leq 1 \ \forall k$
Here: worst case: $k\Delta x = \pi \Rightarrow G(k) = 1 - 4r$
Hence FD scheme conditionally stable:

$\boxed{r \leq \frac{1}{2}}$ (seen before)

Fast version:

$$\frac{G-1}{\Delta t} = D\frac{e^{i\theta}-2+e^{-i\theta}}{(\Delta x)^2} = \frac{2D}{(\Delta x)^2}\cdot(\cos(\theta)-1)$$

$$\Rightarrow G = 1-2r\cdot(1-\cos\theta),\ \theta = k\Delta x$$

Ex.: Crank-Nicolson

$$\frac{U_j^{n+1}-U_j^n}{\Delta t} = D\cdot\frac{1}{2}\cdot\left(\frac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{(\Delta x)^2}+\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{(\Delta x)^2}\right)$$

$$\frac{G-1}{\Delta t} = D\cdot\frac{1}{2}\cdot(G+1)\cdot\frac{e^{i\theta}-2+e^{-i\theta}}{(\Delta x)^2}$$

$$\Rightarrow G = \frac{1-r\cdot(1-\cos\theta)}{1+r\cdot(1-\cos\theta)}$$

Always $|G|\le 1 \Rightarrow$ unconditionally stable.

Ex.: 2D heat equation

$$u_t = u_{xx}+u_{yy}$$

Forward Euler

$$\frac{U_{1j}^{n+1}-U_{ij}^n}{\Delta t} = \frac{U_{i+1,j}^n-2U_{i,j}^n+U_{i-1,j}^n}{(\Delta x)^2}+\frac{U_{i,j+1}^n-2U_{i,j}^n+U_{i,j-1}^n}{(\Delta y)^2}$$

$$u(x,y,t_n) = e^{i(k,l)\cdot\binom{x}{y}} = e^{ikx}\cdot e^{ily}$$

$$\frac{G-1}{\Delta t} = \frac{e^{ik\Delta x}-2+e^{-ik\Delta x}}{(\Delta x)^2}+\frac{e^{il\Delta y}-2+e^{-il\Delta y}}{(\Delta y)^2}$$

$$\Rightarrow G = 1-2\frac{\Delta t}{(\Delta x)^2}\cdot(1-\cos(k\Delta x))-2\frac{\Delta t}{(\Delta y)^2}\cdot(1-\cos(l\Delta y))$$

Worst case: $k\Delta x = \pi = l\Delta y \Rightarrow G = 1-4\frac{\Delta t}{(\Delta x)^2}-4\frac{\Delta t}{(\Delta y)^2}$

Stability condition:

$$\frac{\Delta t}{(\Delta x)^2}+\frac{\Delta t}{(\Delta y)^2}\le\frac{1}{2}\Leftrightarrow \Delta t\le\frac{1}{2}\cdot\left(\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}\right)^{-1}\underbrace{=}_{\text{if } \Delta x=h=\Delta y}\frac{h^2}{4}$$