Ex.: Continuity Equation

$\rho(t) + (v(x)\rho)_x = 0$

$\Leftrightarrow \rho_t + v(x)\rho_x = -v'(x)\rho$

$\Rightarrow$ Characteristic ODE

$$\begin{cases} \dot{x}_j &= v(x_j) \\ \dot{\rho}_j &= -v'(x_j)\rho_j \end{cases}$$

Traffic on road

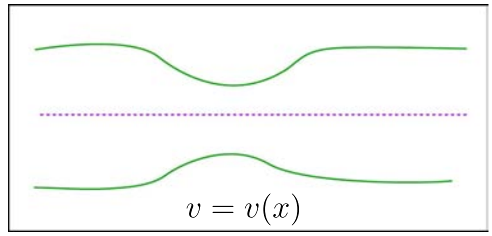


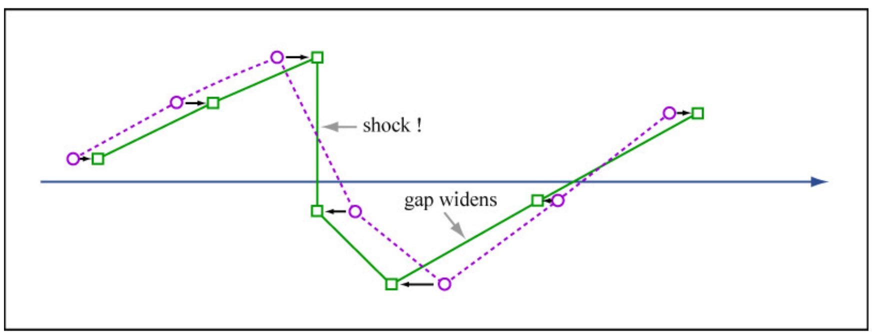
Image by MIT OpenCourseWare.

Ex.: Nonlinear conservation law

$u_t + (\frac{1}{2}u^2)_x = 0$ Burgers' equation

$\Rightarrow \left\{ \begin{array}{l} \dot{x}_j = u_j \\ \dot{u}_j = 0 \end{array} \right\}$ if solution smooth

Problem: Characteristic curves can intersect $\Leftrightarrow$ Particles collide



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Particle management required:

- Merge colliding particles (how?)
- Insert new particles into gaps (where/how?)

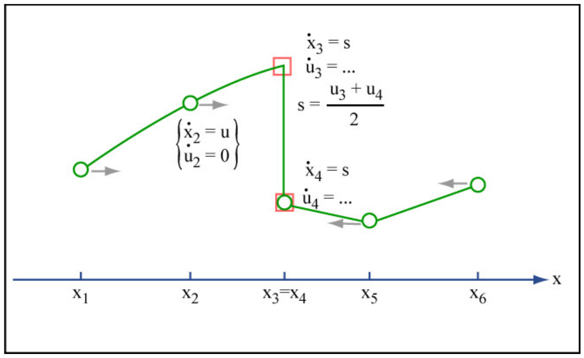
An Exactly Conservative Particle Method for 1D Scalar Conservation Laws

[Farjoun, Seibold JCP 2009]

$u_t + (\frac{1}{2}u^2)_x = 0$

Observe: If we have a piecewise linear function initially, then the exact solution is a piecewise linear function forever (including shocks).

Two choices: (A) Move shock particles using Rankine-Hugoniot condition.



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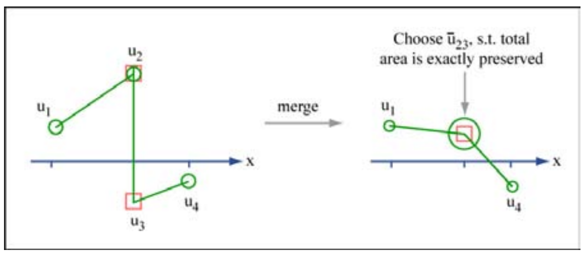
(B) Merge shock particles, then proceed in time.



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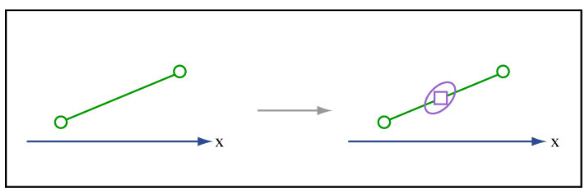
Same for insertion:



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Ex.: Shallow water equations

$$\left\{ \begin{array}{l} h_t + (uh)_x = 0 \\ u_t + uu_x + gh_x = 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \frac{Dh}{Dt} = -u_x h \\ \frac{Du}{Dt} = -gh_x \end{array} \right\}$$

Lagrangian derivative $\frac{Df}{Dt} = f_t + u \cdot f_x$

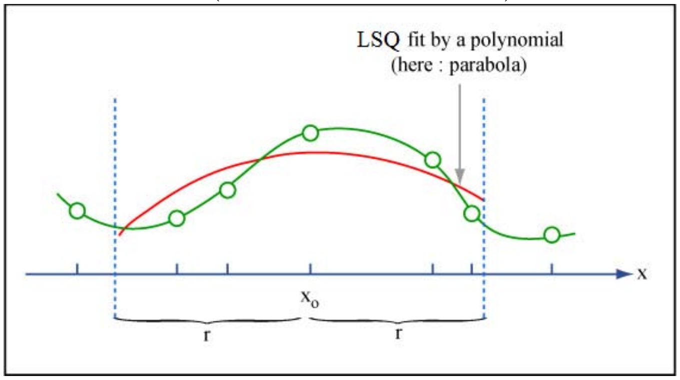
Particle Method:

$$\left\{ \begin{array}{l} \dot{x}_j = u_j \\ \dot{h}_j = -h_j (\partial_x u)(x_j) \\ \dot{u}_j = -g(\partial_x h)(x_j) \end{array} \right\}$$

Required: Approximation to $\partial_x h, \partial_x u$ at x_j

Particles non-equidistant.

Meshfree approximation (moving least squares):



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Local fit: $\hat{u}(x) = ax^2 + bx + c$

Weighted LSQ-fit: $\min_{a,b,c} \sum_{j:|x_j - x_0| \le r} \frac{|\hat{u}(x_j) - u(x_j)|^2}{w(x_j - x_0)}$

$w(x) = \frac{1}{r^\alpha}$ or $= e^{-\alpha r}$ or ...

Define $(\partial_x u)(x_j) = \hat{u}'(x_0)$

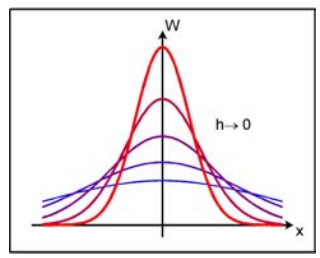
Smoothed Particle Hydrodynamics (SPH)

Quantity f

$f(x) = \int_{\mathbb{R}^d} f(\tilde{x}) \delta(x - \tilde{x}) d\tilde{x}$

Sequence of kernels W^h

$\lim_{h \to 0} W^h(x) = \delta(x)$, $\int_{\mathbb{R}} W^h(x) dx = 1 \ \forall h$.



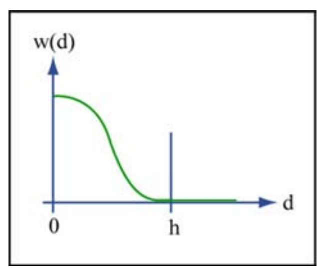


Image by MIT OpenCourseWare.

Also:

$W^h(x) = w_h(||x||)$,

$w(d) = 0 \ \forall d > h \leftarrow$ smoothing length

Approximation I:

$f^h(x) = \int_{\mathbb{R}^d} f(\tilde{x}) W^h(x - \tilde{x}) d\tilde{x}$

$\Rightarrow \lim_{h \to 0} f^h(x) = f(x)$.

Density ρ

Density measure $\mu_\rho(A) = \int_A \rho dx$

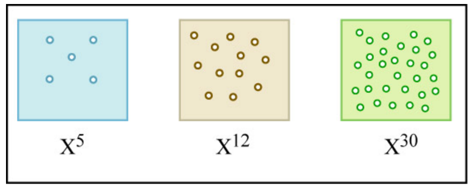
$$f^h(x) = \int_{\mathbb{R}^d} \frac{f(\tilde{x})}{\rho(\tilde{x})} W^h(x - \tilde{x}) \underbrace{\rho(\tilde{x}) d\tilde{x}}_{= d\mu_\rho(\tilde{x})}$$

Sequence of point clouds $\{X^{(n)}\}_{h\in\mathbb{N}}$

$X^{(n)} = (x_1{}^{(n)}, \ldots, x_n{}^{(n)})$

Point measure

$$\delta X^{(m)} = \sum_{i=1}^{n} m_i{}^{(n)} \delta_{x_i^{(n)}} \underset{n\to\infty}{\longrightarrow} \mu_\rho$$



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Approximation II: $f^h(x) \approx \int \frac{f(\tilde{x})}{\rho(\tilde{x})} W^h(x-\tilde{x}) d\delta X^{(n)}(\tilde{x})$

$$= \sum_{i=1}^{n} m_i{}^{(n)} \frac{f(x_i{}^{(n)})}{\rho(x_i{}^{(n)})} W^h(x - x_i{}^{(n)}) =: f^{h,n}(x)$$

$$\Rightarrow \nabla f^{h,n}(x) = \sum_{i=1}^{n} m_i{}^{(n)} \frac{f(x_i{}^{(n)})}{\rho(x_i{}^{(n)})} \underbrace{\nabla W^h}_{\text{hard-code}} (x - x_i{}^{(n)})$$

$$f_k{}^h = \sum_{i=1}^{n} \frac{m_i}{\rho_i} f_i W_{ki}{}^h$$

$$\nabla f_k{}^h = \sum_{i=1}^{n} \frac{m_i}{\rho_i} f_i W_{ki}{}^h$$

Apply to Euler equations of compressible gas dynamics:

$$\begin{array}{l} \text{density} \\ \text{velocity} \\ \text{energy} \end{array} \left\{ \begin{array}{rcl} \frac{\partial \rho}{\partial t} &=& -\nabla\cdot(\rho\vec{u}) \\ \frac{D\vec{u}}{Dt} &=& -\frac{\nabla p}{\rho} \\ \frac{De}{Dt} &=& -\frac{p}{\rho}\nabla\cdot\vec{u} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{rcl} \frac{D\rho}{Dt} &=& \vec{u}\cdot\nabla\rho - \nabla\cdot(\rho\vec{u}) \\ \frac{D\vec{u}}{Dt} &=& -\nabla\left(\frac{p}{\rho}\right) - \left(\frac{p}{\rho^2}\right)\nabla\rho \\ \frac{De}{Dt} &=& u\cdot\nabla\left(\frac{p}{\rho}\right) - \nabla\cdot\left(\frac{p\vec{u}}{\rho}\right) \end{array} \right\}$$

$$\boxed{p = p(s,e)}$$

$$\rightarrow \left\{ \begin{array}{rcl} \dot{\rho_k} &=& \vec{u}_k \sum_i m_i \nabla W_{ki} - \sum_i m_i \vec{u}_i^h \nabla W_{ki} \\ \dot{u_k} &=& -\sum_i \frac{m_i}{\rho_i}\frac{p_i}{\rho_i}\nabla W_{ki} - \frac{p_k}{(\rho_k)^2}\sum_i m_i \nabla W_{ki} \\ \dot{e_k} &=& \vec{u}_k \sum_i \frac{m_i}{\rho_i}\frac{p_i}{\rho_i}\nabla W_{ki} - \sum_i \frac{m_i}{\rho_i}\frac{p_i\vec{v}_i}{\rho_i}\nabla W_{ki} \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{rcl} \dot{\rho_k} &=& \sum_i m_i(\vec{u}_k - \vec{u}_i)\nabla W_{ki} \\ \dot{\vec{u}_k} &=& -\sum_i m_i \left(\frac{\rho_k}{(\rho_k)^2} + \frac{p_i}{(\rho_i)^2}\right)\nabla W_{ki} \\ \dot{e_k} &=& \sum_i m_i \frac{p_i}{(\rho_i)^2}(\vec{u}_k - \vec{u}_i)\nabla W_{ki} \end{array} \right\}$$

$$\text{Since } \frac{d}{dt}\left(\sum_i \rho_i\right) = 0 \Rightarrow \rho_k = \sum_i m_i W_{ki}$$

SPH Approximation to Euler Equations:

$$\dot{m}_k = 0\,(\text{or } \neq 0 \text{ if adaptive})$$

$$\dot{\vec{x}}_k = \vec{u}_k$$

$$\dot{\vec{u}}_k = -\sum_i m_i \left(\frac{p_k}{(\rho_k)^2} + \frac{p_i}{(\rho_i)^2}\right) \nabla W_{ki}$$

$$\dot{e}_k = \sum_i m_i \frac{p_i}{(\rho_i)^2} (\vec{u}_k - \vec{u}_i) \nabla W_{ki}$$

$$\rho_k = \sum_i m_i W_{ki}$$

$$p_k = p(\rho_k, e_k)$$