Heat equation

$u_t = \nabla^2 u$

Physics:

Fick's law: flux $F = -a\nabla u$

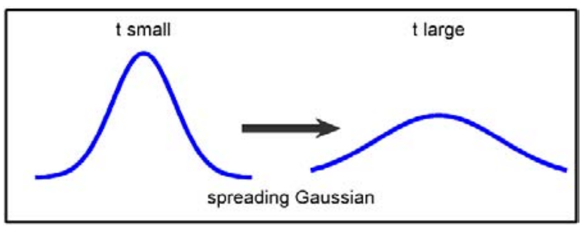
mass balance: $\frac{d}{dt}\int_V u\,dx = -b\int_{\partial V} F\cdot n\,dS = -b\int_V \mathrm{div}F\,dx$

$\Rightarrow u_t = -b\,\mathrm{div}(-a\nabla u) = \underset{\substack{\uparrow\\ \text{simple: } c=1}}{c}\,\nabla^2 u$

Fundamental Solution

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

solves $\left\{ \begin{array}{ll} u_t = \nabla^2 u & \text{in } \mathbb{R}^n \times\,]\,0,\infty\,[\\ u(x,0) = \delta(x) & t = 0 \end{array} \right\}$



Superposition: $\quad\quad \Phi \in C^\infty$

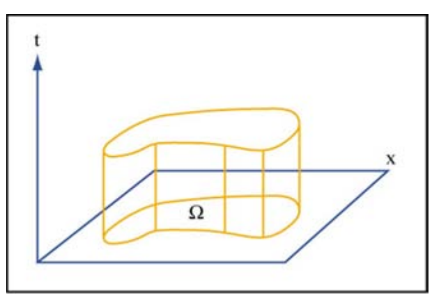
$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)u_0(y)\,dy \quad\quad \Downarrow$

solves $\left\{ \begin{array}{ll} u_t = \nabla^2 u & \text{in } \mathbb{R}^n \times\,]\,0,\infty\,[\\ u(x,0) = u_0(x) & \end{array} \right\} \quad\quad u \in C^\infty$

Maximum Principle

$\Omega \in \mathbb{R}^n$ bounded

$\Omega_T := \Omega \times [0,T],\ \partial\Omega_T = (\Omega \times \{0\}) \cup (\partial\Omega \times\ ]\,0,T\,[\,)$



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If u is the solution to

$$\left\{ \begin{array}{ll} u_t = \nabla^2 u & \text{in } \Omega_T \\ u = u_0 & \text{on } \Omega \times \{0\} \\ u = g & \text{on } \partial\Omega \times \left] \, 0, T \, \right[\end{array} \right\}$$

then

(i) $\max\limits_{\overline{\Omega}_T} u = \max\limits_{\partial\Omega_T} u$ (weak)

(ii) For Ω is connected:

If $\exists (x_0, t_0) \in \Omega_T : u(x_0, t_0) = \max\limits_{\overline{\Omega}_T} u$, then $u = \text{constant}$ (strong)

<u>Implications</u>:

- max $\to$ min
- uniqueness (see Poisson equation)
- infinite speed of propagation:

$$\left\{ \begin{array}{ll} u_t = \nabla^2 u & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times \left] \, 0, T \, \right[\\ u = g & \text{on } \Omega \times \{0\} \end{array} \right\}$$

$$g \geq 0 \overset{\substack{\text{strong max principle} \\ \downarrow}}{\Longrightarrow} u > 0 \text{ in } \Omega_T.$$

<u>Inhomogenous Case</u>:

$$\left\{ \begin{array}{ll} u_t - \nabla^2 u = f & \text{in } \mathbb{R}^n \times \left] \, 0, \infty \, \right[\\ u = 0 & \text{on } \mathbb{R}^n \times \{0\} \end{array} \right\}$$

solution: $u(x,t) = \displaystyle\int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) \, dy \, ds$

"Duhamel's Principle" (variation of constants):
superposition of solutions starting at s with initial conditions $f(s)$.

<u>Transport equation</u>

$$\left\{ \begin{array}{ll} u_t + b \cdot \nabla u = 0 & \text{in } \mathbb{R}^n \times \left] \, 0, \infty \, \right[\\ u = u_0 & \text{on } \mathbb{R}^n \times \{0\} \end{array} \right\} \qquad b = \text{direction vector (field)}$$

solution: $u(x,t) = u_0(x - tb)$.
check: $u_t = -b \cdot \nabla u_0(x - tb) = -b \cdot \nabla u$ ✓

<u>Inhomogenous Case</u>:

$$\left\{ \begin{array}{ll} u_t + b \cdot \nabla u = f & \text{in } \mathbb{R}^n \times \left] \, 0, \infty \, \right[\\ u = u_0 & \text{on } \mathbb{R}^n \times \{0\} \end{array} \right\}$$

Duhamel's principle yields the solution:

$$u(x,t) = u_0(x - tb) + \int_0^t f(x + (s-t)b, s) \, ds$$

Wave equation

$$u_{tt} - \nabla^2 u = \underbrace{f}_{\text{source}}$$

$\boxed{\text{1D}}$ $\left\{ \begin{array}{ll} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}^n \times]\,0,\infty\,[\\ u = g, u_t = h & \text{on } \mathbb{R}^n \times \{0\} \end{array} \right\}$

$0 = u_{tt} - u_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x)u$

Define: $v(x,t) := (\partial_t - \partial_x)u(x,t)$

$\Rightarrow v_t + v_x = 0 \overset{\text{transport}}{\Rightarrow} v(x,t) = a(x-t)$

Thus: $u_t - u_x = a(x-t)$

(inhomogenous transport) $[b = -1, f(x,t) = a(x-t)]$

$$\Rightarrow u(x,t) = \int_0^t a(x + (t-s) - s)\,ds + b(x+t)$$

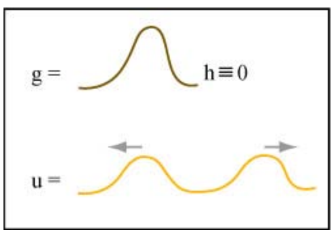
$$= \tfrac{1}{2}\int_{x-t}^{x+t} a(y)\,dy + b(x+t)$$

initial conditions: $b = g, a = h - g_x$

$$\Rightarrow \boxed{u(x,t) = \tfrac{1}{2}(g(x+t) + g(x-t)) + \tfrac{1}{2}\int_{x-t}^{x+t} h(y)\,dy}$$

d'Alembert's formula.

Ex.:



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Higher space dimensions

$\left\{ \begin{array}{ll} u_{tt} - \nabla^2 u = 0 & \text{in } \mathbb{R}^n \times]\,0,\infty\,[\\ u = g, u_t = h & \text{on } \mathbb{R}^n \times \{0\} \end{array} \right\}$

$\boxed{\text{3D}}$ $u(x,t) = \int_{\partial B(x,t)} th(y) + g(y) + \nabla g(y) \cdot (y-x)\,dS(y)$

Kirchhoff's formula.

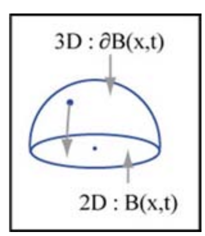
$\boxed{\text{2D}}$ Obtain from 3D solution by projecting to 2D

$\bar{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t)$

yields

$$u(x,t) = \tfrac{1}{2}\int_{B(x,t)} \frac{tg(y) + t^2 h(y) + t\nabla g(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}}\,dy$$

Poisson's formula.



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Ex.: $g \equiv 0, h = \delta$.

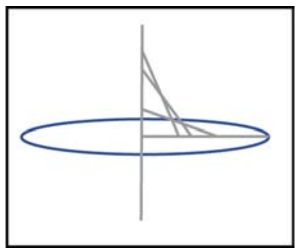
3D $u(x,t) = t \int_{\partial B(x,t)} \delta(y)\, dy = \left\{ \begin{array}{cc} \frac{1}{4\pi t} & |x| = t \\ 0 & \text{else} \end{array} \right\}$

sharp front

2D $u(x,t) = \frac{1}{2} \int_{B(x,t)} \frac{t^2}{(t^2 - |y-x|^2)^{1/2}} \cdot \delta(y)\, dy = \frac{1}{2\pi t^2} \cdot \frac{t^2}{(t^2 - |x|^2)^{1/2}}$

$$= \left\{ \begin{array}{cc} \frac{1}{2\pi (t^2 - |x|^2)^{1/2}} & x \le t \\ 0 & \text{else} \end{array} \right\}$$

signal never vanishes



18.336 Numerical Methods for Partial Differential Equations
Spring 2009