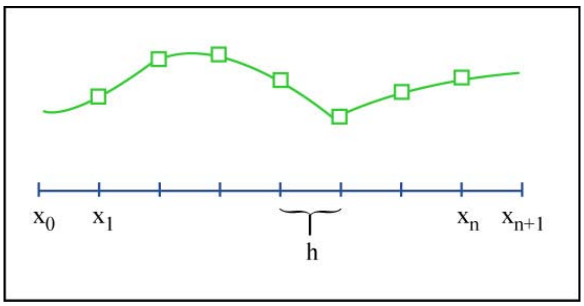
General Linear Second Order Equation

$$\left\{ \begin{array}{l} \underbrace{a(x)u_{xx}(x)}_{\text{diffusion}} + \underbrace{b(x)u_x(x)}_{\text{advection}} + \underbrace{c(x)u(x)}_{\text{growth/decay}} = \underbrace{f(x)}_{\text{source}} \quad x \in]0,1[\\ u(0) = \alpha \\ u(1) = \beta \end{array} \right\}$$

Approximation:

$$a_i \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + b_i \frac{u_{i+1} - u_{i-1}}{2h} + c_i u_i = f_i$$

where $a_i = a(x_i), b_i = b(x_i), c_i = c(x_i)$.



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Linear system: $A \cdot \vec{u} = \vec{f}$

$$A = \frac{1}{h^2} \begin{bmatrix} h^2c_1 - 2a_1 & a_1 + \frac{hb_1}{2} & & \\ a_2 - \frac{hb_2}{2} & h^2c_2 - 2a_2 & a_2 + \frac{hb_2}{2} & \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & a_{n-1} - \frac{hb_{n-1}}{2} & h^2c_{n-1} - 2a_{n-1} & a_{n-1} + \frac{hb_{n-1}}{2} \\ & & a_n - \frac{hb_n}{2} & h^2c_n - 2a_n \end{bmatrix}$$

$$\vec{f} = \begin{bmatrix} f_1 - \left(\frac{a_1}{h^2} - \frac{b_1}{2h}\right)\alpha \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n - \left(\frac{a_n}{h^2} + \frac{b_n}{2h}\right)\beta \end{bmatrix}$$

Potential Problems:

- A non-symmetric
- If $|a(x)| \ll |b(x)|$, instabilities possible due to central differences.

Often better approximations possible.

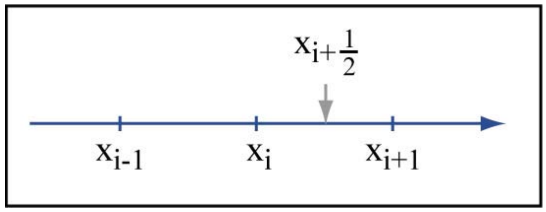
Ex.: Heat equation in rod with variable conductivity

$$(\kappa(x)u_x)_x = f(x) \qquad (*)$$

$$\Leftrightarrow \kappa(x)u_{xx} + \kappa_x(x)u_x = f(x) \quad (**)$$

Can discretize $(**)$ as before. Suboptimal results.
Better: discretize $(*)$ directly (in line with physics)

Use cell centers



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$$\kappa(x_{i+\frac{1}{2}})u_x(x_{i+\frac{1}{2}}) \approx \kappa_{i+\frac{1}{2}} \cdot \frac{u_{i+1}-u_i}{h}$$

$$\Rightarrow (\kappa u_x)_x(x_i) \approx \frac{1}{h}\left(\kappa_{i+\frac{1}{2}} \cdot \frac{u_{i+1}-u_i}{h} - \kappa_{i-\frac{1}{2}} \cdot \frac{u_i - u_{i-1}}{h}\right)$$

$$= \frac{1}{h}(\kappa_{i-\frac{1}{2}}u_{i-1} - (\kappa_{i-\frac{1}{2}} + \kappa_{i+\frac{1}{2}})u_i + \kappa_{i+\frac{1}{2}}u_{i+1})$$

$$A = \frac{1}{h^2}\begin{bmatrix} -(\kappa_{\frac{1}{2}}+\kappa_{\frac{3}{2}}) & \kappa_{\frac{3}{2}} & & & \\ \kappa_{\frac{3}{2}} & -(\kappa_{\frac{3}{2}}+\kappa_{\frac{5}{2}}) & \kappa_{\frac{5}{2}} & & \\ & \ddots & \ddots & \ddots & \\ & & \kappa_{n-\frac{3}{2}} & -(\kappa_{n-\frac{3}{2}}+\kappa_{n-\frac{1}{2}}) & \kappa_{n-\frac{1}{2}} \\ & & & \kappa_{n-\frac{3}{2}} & -(\kappa_{n-\frac{1}{2}}+\kappa_{n+\frac{1}{2}}) \end{bmatrix}$$

Symmetric matrix, $-A$ positive definite.

Great for linear solvers $\longrightarrow$ CG (conjugate gradient method).

2D/3D

$$\nabla \cdot (\kappa \nabla u) = f$$

$$||$$

2D: $(\kappa(x)u_x)_x + (\kappa(x)u_y)_y \longleftarrow 2\times$ 1D

Errors, Consistency, Stability

Presentation for Poisson equation, but results transfer to any linear finite difference scheme for linear PDE.

$$u''(x) = f(x) \rightsquigarrow A \cdot \underset{\substack{\uparrow \\ \text{vector of approximate function values } U_i}}{U} = F$$

true solution values: $\hat{U} = \begin{bmatrix} u(x_1) \\ \vdots \\ u(x_n) \end{bmatrix}$

Local Truncation Error (LTE)

Plug true solution $u(x)$ into FD scheme:

$$\begin{aligned} \tau_i &= \tfrac{1}{h^2}(u(x_{i-1}) - 2u(x_i) + u(x_{i+1})) - f(x_i) \\ &= u''(x_i) + \tfrac{1}{12}u''''(x_i)h^2 + O(h^4) - f(x_i) \\ &= \tfrac{1}{12}u''''(x_i)h^2 + O(h^4) \end{aligned}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} = A \cdot \hat{U} - F$$

$$\Rightarrow A\hat{U} = F + \tau$$

Global Truncation Error (GTE)

Error vector: $E : U - \hat{U}$

$$\left.\begin{aligned} AU &= F \\ A\hat{U} &= F + \tau \end{aligned}\right\} \Rightarrow AE = -\tau \text{ and } E = 0 \text{ at boundaries}$$

Discretization of $\left\{ \begin{array}{ll} -e''(x) = -\tau(x) &]0,1[\\ e(0) = 0 = e(1) & \end{array} \right\}$

$T(x) \approx \frac{1}{12}u''''(x)h^2$

$\Rightarrow e(x) \approx -\frac{1}{12}u''(x)h^2 + \frac{1}{12}h^2(u''(0) + x(u''(1) - u''(0))$

Message: Global error order = local error order if method stable.

Stability

Mesh size $h : A^h \cdot E^h = -\tau^h$

$\Rightarrow E^h = -(A^h)^{-1} \cdot \tau^h$

$\Rightarrow ||E^h|| = ||(A^h)^{-1} \cdot \tau^h|| \leq ||(A^h)^{-1}|| \cdot \underbrace{||\tau^h||}_{O(h^2)(\text{LTE})}$

Stability: $||(A^h)^{-1}|| \leq C \quad \forall\, h < h_0$
Inverse FD operators uniformly bounded.

$\Rightarrow ||E^h|| \leq C \cdot ||\tau^h|| \;\forall\; h < h_0.$

Consistency

$||\tau^h|| \to 0$ as $h \to 0$
LTE goes to 0 with mesh size

Convergence

$||E^h|| \to 0$ as $h \to 0$
GTE goes to 0 with mesh size

Lax Equivalence Theorem

consistency + stability $\Longleftrightarrow$ convergence

Proof: (only "$\Longrightarrow$" here)

$$||E^h|| \leq ||(A^h)^{-1}|| \cdot ||\tau^h|| \underset{\substack{\uparrow \\ \text{stability}}}{\leq} C \cdot \underset{\substack{\uparrow \\ \text{consistency}}}{||\tau^h||} \longrightarrow 0 \text{ as } h \to 0$$

Also: $O(h^P)$ LTE + stability $\Longrightarrow O(h^P)$ GTE

Stability for Poisson Equation

Consider 2-norm
$||U||_2 = \left(\sum_i {U_i}^2\right)^{\frac{1}{2}}$
$||A||_2 = \rho(A) = \max_p|\lambda_p|$ largest eigenvalue
$\Rightarrow ||A^{-1}||_2 = \rho(A^{-1}) = \max_p|{\lambda_p}^{-1}| = (\min_p|\lambda_p|)^{-1}$
Stable, if eigenvalues of A^h bounded away from 0 as $h \to 0$
In general, difficult to show.
But for Poisson equation with Dirichlet boundary conditions, it is known that

$\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1)$
$\Rightarrow \lambda_1 = \frac{2}{h^2}(-\frac{1}{2}\pi^2 h^2 + O(h^4)) = -\pi^2 + O(h^2) \qquad$ Stable ✓
Hence: $||E^h||_2 \leq ||(A^h)^{-1}||_2 \cdot ||\tau^h||_2 \approx \frac{1}{\pi^2}||\tau^h||_2.$