

1 Math basics

1.1 Derivatives and differential equations

In this course, we will mostly deal with ordinary differential equations (ODEs) and partial differential equations (PDEs) real-valued scalar or vector fields. Usually, non-bold symbols will be reserved for scalar objects f (e.g., mass density) and bold font $\mathbf{f} = (f_1, f_2, \dots)$ for vector-valued objects, such as time-dependent position vectors $\mathbf{x}(t)$ or velocity fields $\mathbf{v}(t, \mathbf{x}) = (v_1(t, \mathbf{x}), v_2(t, \mathbf{x}), \dots)$.

ODEs are equations that contain derivatives of scalar or vector-valued or, more generally, tensor-valued functions $f(x)$ of a single variable x . Depending on context, we will denote derivatives of such functions by

$$\frac{df}{dt} = \dot{f} = f' = f_x \quad (1)$$

PDEs are equations that contain derivatives of scalar or vector-valued functions $f(x_1, x_2, \dots)$ of more than one variable. Depending on context, we will denote partial derivatives by

$$\frac{\partial f}{\partial x_i} = \partial_{x_i} f = \partial_i f = f_{x_i} = f_{,i} \quad (2)$$

In standard 3D Cartesian coordinates (x_1, x_2, x_3) defined with respect to some global orthonormal frame Σ , spanned by the basis vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, the gradient-operator ∇ is defined by

$$\nabla = \partial_x \mathbf{e}_1 + \partial_y \mathbf{e}_2 + \partial_z \mathbf{e}_3 = \sum_{i=1}^3 \partial_i \mathbf{e}_i \equiv \partial_i \mathbf{e}_i, \quad (3)$$

where we have introduced the Einstein summation convention on the rhs. Applying ∇ to a scalar function f gives a vector

$$\nabla f = (\partial_1 f, \partial_2 f, \dots) \quad (4)$$

whereas application of ∇ to a tensorial quantity depends on the choice of the product: For instance, in the case of a 3D vector field $\mathbf{v}(t, \mathbf{x})$, we can obtain a scalar field called *divergence*

$$\nabla \cdot \mathbf{v} \equiv \partial_i v_i, \quad (5a)$$

another (pseudo-)vector field called *curl*

$$\nabla \wedge \mathbf{v} \equiv (\epsilon_{ijk} \partial_j v_k) \quad (5b)$$

and the *gradient* matrix

$$\nabla \mathbf{v} \equiv (\partial_i v_j). \quad (5c)$$

The (scalar) Laplacian operator Δ in Cartesian coordinates is defined by

$$\Delta \equiv \nabla^2 \equiv \sum_i \partial_i \partial_i = \partial_{ii}. \quad (6)$$

Please recall how these operators look in cylindrical and spherical coordinates.

1.2 Linear and nonlinear objects

A mathematical operator - or, more generally, some property - \mathcal{P} defined on vectors or functions f, g , is said to be *linear*, if it satisfies

$$\mathcal{P}(\alpha f + \beta g) = \alpha \mathcal{P}(f) + \beta \mathcal{P}(g) \quad (7)$$

Important examples are derivatives, integrals and expectation values.

A famous linear ODE, is the simple harmonic oscillator equation

$$\ddot{x}(t) = -\omega^2 x(t) \quad (8)$$

which has fundamental sin- and cos-solutions, that can be used to construct more general solutions by superposition.

An important (homogeneous) linear PDE, is Laplace's equation

$$\nabla^2 f(t, \mathbf{x}) = 0. \quad (9)$$

Functions f satisfying this equation are called *harmonic*.

Later on, we will often try to approximate nonlinear PDEs through linear PDEs.

1.3 Complex numbers and functions

Although we will mostly deal with real fields in this course, it is sometimes helpful to rewrite equations in terms of complex quantities, especially, when dealing with 2D hydrodynamic problems. Complex numbers are 2D extensions of real numbers,

$$z = x + iy \in \mathbb{C}, \quad i^2 = -1 \quad (10)$$

with real part $\Re z = x \in \mathbb{R}$ and imaginary part $\Im z = y \in \mathbb{R}$. The complex conjugate of a real number is given by

$$\bar{z} = x - iy \quad (11)$$

and corresponds to a reflection at the real axis or, equivalently, at the line $\Im(z) = 0$.

Addition of complex numbers is linear

$$z = z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) = x + iy \quad (12)$$

corresponding to the addition of the two 2D vectors (x_1, y_1) and (x_2, y_2) . In contrast, complex multiplication mixes real and imaginary parts

$$z = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) = x + iy \quad (13)$$

A complex function is a map

$$f : \mathbb{C} \rightarrow \mathbb{C}, \quad f(z) = (u(x, y), v(x, y)) \quad (14)$$

that can be interpreted as a map from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. A function f is said to be complex differentiable (or *analytic* or *holomorphic*) if it satisfies the Cauchy-Riemann equations

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v \quad (15a)$$

By differentiating again, we find

$$\partial_x^2 u = \partial_x \partial_y v = -\partial_y^2 u = 0 \quad (15b)$$

$$\partial_y^2 v = \partial_y \partial_x u = -\partial_x^2 v = 0; \quad (15c)$$

$$(15d)$$

this means that analytic functions $f = (u, v)$ are harmonic

$$\nabla^2 f = 0. \quad (15e)$$

An analytic function that we will frequently encounter is the exponential function

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \dots \quad (16)$$

Euler's formula

$$e^{i\phi} = \cos \phi + i \sin \phi, \quad \phi \in \mathbb{R} \quad (17)$$

relates exp to the trigonometric sin-and cos-functions.

When dealing with axisymmetric problems it is often advantageous to use the polar representation of a complex number

$$z = r e^{i\phi}, \quad r = |z| = \sqrt{z\bar{z}} \in \mathbb{R}_0^+, \quad \phi = \arctan 2(y, x) \in [0, 2\pi) \quad (18)$$

From the properties of the exp-function, it follows that the multiplication of complex numbers

$$z = z_1 z_2 = r_1 e^{i\phi_1} r_2 e^{i\phi_2} = r_1 r_2 e^{i(\phi_1 + \phi_2)} \quad (19)$$

corresponds to a combined rotation and dilatation.

1.4 Fourier transforms

The Fourier transform of a function $f(t)$ is defined by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} f(t), \quad (20a)$$

its the inverse is given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \hat{f}(\omega) \quad (20b)$$

In particular, for the Dirac delta-function $\delta(t)$

$$\hat{\delta}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} \delta(t) = \frac{1}{\sqrt{2\pi}}, \quad (21)$$

yielding the useful Fourier representation

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \quad (22)$$

These definitions and properties extend directly to higher dimensions.

A main advantage of Fourier transformations is that they translate differential equations into simpler algebraic equations.

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