19 Stream functions and conformal maps

There is a useful device for thinking about two dimensional flows, called the *stream function* of the flow. The stream function $\psi(x, y)$ is defined as follows

$$\boldsymbol{u} = (u, v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right). \tag{469}$$

The velocity field described by ψ automatically satisfies the incompressibility condition, and it should be noted that

$$\boldsymbol{u} \cdot \nabla \psi = u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} = 0. \tag{470}$$

Thus ψ is constant along streamlines of the flow. Besides it's physical convenience, another great thing about the stream function is the following. By definition

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}, \tag{471a}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}, \tag{471b}$$

where ϕ is the velocity potential for an irrotational flow. Thus, both ϕ and ψ obey the well known *Cauchy-Riemann* equations of complex analysis.

19.1 The Cauchy-Riemann equations

In complex analysis you work with the complex variable $z = x + iy$. Thus, if you have some complex function $f(z)$ what is df/dz? Well, $f(z)$ can be separated into a real part $u(x, y)$ and an imaginary part $v(x, y)$, where u and v are real functions, i.e.:

$$f(z) = f(x + iy) = u(x, y) + iv(x, y). \tag{472}$$

For example, if $f(z) = z^2$ then $u = x^2 - y^2$ and $v = 2xy$. What then is df/dz? Since we are now in two-dimensions we can approach a particular point z from the x-direction or the y-direction (or any other direction, for that matter). On one hand we could define

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{473a}$$

Or, alternatively

$$\frac{df}{dz} = \frac{\partial f}{\partial (iy)} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{473b}$$

For the definition of the derivative to make sense requires $\partial u/\partial x = \partial v/\partial y$ and $-\partial u/\partial y = \partial v/\partial x$, the Cauchy-Riemann equations. If this is true then $f(z)$ is said to be analytic and we can simply differentiate with respect to z in the usual manner. For our simple example $f(z) = z^2$ we have that $df/dz = 2z$ (confirm for yourself that z^2 is analytic as there are many functions that are not, e.g., $|z|$ is not an analytic function.)

19.2 Conformal mapping

We can now use the power of complex analysis to think about two dimensional potential flow problems. Since ϕ and ψ obey the Cauchy-Riemann equations, this implies that $w = \phi + i\psi$ is an analytic function of the complex variable $z = x + iy$. We call w the *complex potential*. Another important property of 2D incompressible flow is that both ϕ and ψ satisfy Laplace's equation. For example, using the Cauchy-Riemann equations we see that

$$\frac{\partial \psi}{\partial x^2} + \frac{\partial \psi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0. \tag{474}$$

The same proof can be used for ϕ. We can therefore consider any analytic function (e.g., $\sin z$,z^4,...), calculate the real and imaginary parts and both of them satisfy Laplace's equation.

The velocity components u and v are directly related to dw/dz, which is conveniently calculated as follows:

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv. \tag{475}$$

As a simple example consider uniform flow at an angle α to the x-axis. The corresponding complex potential is $w = u_0 z e^{-i\alpha}$. In this case $dw/dz = u_0 e^{-i\alpha}$. Using the above relation, this tells us that $u = u_0 \cos\alpha$ and $v = u_0 \sin\alpha$.

We can also determine the complex potential for flow past a cylinder since we know that

$$\phi = u_0 \left(r + \frac{R^2}{r} \right) \cos\theta, \tag{476}$$

and this is just the real part of the complex potential

$$w = u_0 \left(z + \frac{R^2}{z} \right). \tag{477}$$

Check this by substituting in $z = re^{i\theta}$. What is the corresponding stream function? Also $w(z) = -i \ln z$ is the complex potential for a point vortex since

$$Re\,(w(z)) = Re\left(-i \ln(re^{i\theta}) \right) = \theta, \tag{478}$$

and we know that $\phi = \theta$ is the real potential for a point vortex. Thus

$$w(z) = u_0 \left(z + \frac{R^2}{z} \right) - \frac{i\Gamma}{2\pi} \ln z \tag{479}$$

is the complex potential for flow past a cylinder with circulation Γ.

So let's assume that the only problem we know how to solve is flow past a cylinder, when really we want to know how to solve for flow past an aerofoil. The idea is to now consider two complex planes (x, y) and (X, Y). In the first plane we have the complex variable $z = x + iy$ and in the latter we have $Z = X + iY$. If we construct a mapping $Z = F(z)$ which is analytic, with an inverse $z = F^{-1}(Z)$, then $W(Z) = w(F^{-1}(Z))$ is also analytic, and may be considered a complex potential in the new co-ordinate system. Because $W(Z)$ and $w(z)$ take the same value at corresponding points of the two planes it follows that Ψ and ψ are the same at corresponding points. Thus streamlines are mapped into streamlines. In particular a solid boundary in the z-plane, which is necessarily a streamline, gets mapped into a streamline in the Z-plane, which could accordingly be viewed as a rigid boundary. Thus all we have done is distort the streamlines and the boundary leaving us with the key question: Given flow past a circular cylinder in the z-plane can we choose a mapping so as to obtain in the Z-plane uniform flow past a more wing-like shape? (Note that we have brushed passed some technical details here, such as the requirement that $dF/dz \neq 0$ at any point, as this would cause a blow-up of the velocity).

19.3 Simple conformal maps

The simplest map is

$$Z = F(z) = z + b, \tag{480}$$

which corresponds to a translation. Then there is

$$Z = F(z) = ze^{i\alpha}, \tag{481}$$

which corresponds to a rotation through angle α. In this case, the complex potential for uniform flow past a cylinder making angle α with the stream is

$$W(Z) = u_0 \left(Ze^{-i\alpha} + \frac{R^2}{Z} e^{i\alpha} \right) - \frac{i\Gamma}{2\pi} \ln Z. \tag{482}$$

Note, that this expression could also include the term $\ln e^{i\alpha} = i\alpha$ which I have neglected. This is just a constant however and doesn't change the velocity.

Finally there is the non-trivial *Joukowski transformation*,

$$Z = F(z) = z + \frac{c^2}{z}. \tag{483}$$

What does this do to the circle? Well, $z = ae^{i\theta}$ becomes

$$Z = ae^{i\theta} + \frac{c^2}{a} e^{-i\theta} = (a + \frac{c^2}{a}) \cos\theta + i(a - \frac{c^2}{a}) \sin\theta. \tag{484}$$

Defining $X = Re(Z)$, $Y = Im(Z)$, it is easily shown that

$$\left(\frac{X}{a + \frac{c^2}{a}} \right)^2 + \left(\frac{Y}{a - \frac{c^2}{a}} \right)^2 = 1, \tag{485}$$

which is the equation of an ellipse, provided $c < a$.