20 Classical aerofoil theory

We now know that through conformal mapping it is possible to transform a circular wing into a more realistic shape, with the bonus of also getting the corresponding inviscid, irrotational flow field. Let's consider some more realistic shapes and see what we get.

20.1 An elliptical wing

First let's rotate our cylinder by an angle α. The complex potential becomes

$$w(z) = u_0 \left(z e^{-i\alpha} + \frac{R^2}{z} e^{i\alpha} \right) - \frac{i\Gamma}{2\pi} \ln z. \tag{486}$$

Now, using the Joukowski transformation we want to turn our circular wing into an elliptical wing. The transformation stipulates that $Z = z + c^2/z$, so that

$$z_{\pm}(Z) = \frac{Z}{2} \pm \sqrt{\frac{Z^2}{4} - c^2}. \tag{487}$$

Considering $z_+(Z)$

$$w(z) = u_0 \left[z_+(Z) e^{-i\alpha} + \frac{R^2}{z_+(Z)} e^{i\alpha} \right] - \frac{i\Gamma}{2\pi} \ln z_+(Z). \tag{488}$$

If we choose $c = R$, then the ellipse collapses to a flat plate. The velocity components in the Z plane are

$$U - iV = \frac{dW}{dZ} = u_0 \cos\alpha - \frac{i(\Gamma + 2\pi u_0 Z \sin\alpha)}{2\pi\sqrt{Z^2 - 4R^2}}, \tag{489}$$

which can also be written as

$$U - iV = \frac{dw/dz}{dZ/dz} = \frac{u_0 \left(e^{-i\alpha} - \frac{e^{i\alpha}R^2}{z^2} \right) - \frac{i\Gamma}{2\pi z}}{1 - R^2/z^2} \tag{490}$$

On the surface of the body we have $z = Re^{i\theta}$, so the velocities become

$$U - iV = \frac{u_0 \left(e^{-i\alpha} - e^{-2i\theta} e^{i\alpha} \right) - \frac{i\Gamma e^{-i\theta}}{2\pi R}}{1 - e^{-2i\theta}}. \tag{491}$$

At $\theta = 0$ and $\theta = \pi$ we are in trouble because the velocities are infinite. Notably, however, this problem can be removed at $\theta = 0$ if the circulation is chosen so that the numerator vanishes

$$U - iV = \frac{e^{-i\theta} \left[u_0 (e^{i(\theta-\alpha)} - e^{-i(\theta-\alpha)}) - \frac{i\Gamma}{2\pi R} \right]}{1 - e^{-2i\theta}}. \tag{492}$$

Thus for a finite velocity at $\theta = 0$ we require

$$u_0(e^{-i\alpha} - e^{i\alpha}) - \frac{i\Gamma}{2\pi R} = 0, \tag{493}$$

giving

$$\Gamma = -4\pi u_0 R \sin\alpha. \tag{494}$$

In this case flow leaves the trailing edge smoothly and parallel to the plate. Note that it is not possible to cancel out singularities at both ends simultaneously, as we have to rotate in the opposite direction to cancel out the singularity at $\theta = \pi$.

20.2 Flow past an aerofoil

What if we could now construct a mapping with a singularity just on one side ? This we can do by considering a shifted circle, that passes through $z = R$ but encloses $z = -R$. In this case we obtain an aerofoil with a rounded nose but a sharp trailing edge. The boundary of the appropriate circle is prescribed by

$$z = -\lambda + (a + \lambda) e^{i\theta}, \tag{495}$$

where θ is a parameter. First we must modify the complex potential for flow past a cylinder to take account of this new geometry. We have that

$$w(z) = u_0 \left[(z + \lambda) e^{-i\alpha} + \frac{(R + \lambda)^2}{(z + \lambda)^2} e^{i\alpha} \right] - \frac{i\Gamma}{2\pi} \ln(z + \lambda). \tag{496}$$

To find the complex potential for the aerofoil one must then substitute in $z = Z/2 + \sqrt{Z^2/4 - R^2}$. Determining the velocities as before we find that

$$u - iv = \frac{dW}{dZ} = \frac{dw/dz}{dZ/dz} = u_0 \frac{e^{-i\alpha} - \left(\frac{R+\lambda}{z+\lambda}\right)^2 - \frac{i\Gamma}{2\pi(z+\lambda)}}{1 - \frac{R^2}{z^2}}. \tag{497}$$

The value of Γ that makes the numerator zero at the trailing edge is

$$\Gamma = -4\pi u_0 (R + \lambda) \sin\alpha. \tag{498}$$

The flow is then smooth and free of singularities everywhere (because we have successfully trapped the rogue singularity inside the wing), and this is an example of the *Kutta-Joukowski condition* at work.

Our goal in the reminder of this part is to show that our earlier results $F_L = \rho\ell u_0 \Gamma$ and $F_D = 0$ are unaffected by the wing shape. To this end, we first derive Blasius' lemma and then the Kutta-Joukowski theorem.

20.3 Blasius' lemma

To derive Blasius' lemma, we consider the force acting per unit length on the wing, $\boldsymbol{f} = \boldsymbol{F}/\ell = (f_x, f_y)$, which is obtained by integrating the pressure over the (now arbitrary) surface contour ∂S

$$\boldsymbol{f} = -\oint_{\partial S} p\,\boldsymbol{n}\,ds \tag{499}$$

where $\boldsymbol{n}$ is the outward surface normal vector and ds the arc length. Denote by $dz = dx + idy$ a small change along the curve ∂S. In complex notation, the normal element $\boldsymbol{n}\,ds$ can then be expressed as

$$-idz = dy - idx, \tag{500}$$

and Eq. (499) can be rewritten as

$$f := f_x + i f_y = i \oint p\,dz. \tag{501}$$

From Bernoulli we have that $p = p_0 - \rho|v|^2/2$, where

$$v = v_x - i v_y = \frac{dw}{dz}\,, \qquad |v|^2 = v\bar{v} = v_x^2 + v_y^2 = \left|\frac{dw}{dz}\right|^2. \tag{502}$$

Thus, we find

$$f = i \oint \left(p_0 - \frac{\rho}{2}|v|^2\right)\,dz = -i\frac{\rho}{2}\oint |v|^2\,dz \tag{503}$$

Taking the complex conjugate, we have

$$\bar{f} = f_x - i f_y = i\frac{\rho}{2}\oint |v|^2\,d\bar{z}. \tag{504}$$

Furthermore, since v is parallel to z on boundary (which is a stream line), we may use

$$0 = v_x dy - v_y dx \tag{505}$$

to rewrite

$$\begin{aligned}
v^2\, dz &= (v_x - iv_y)^2 (dx + idy) \\
&= (v_x^2 - v_y^2 - 2iv_x v_y)(dx + idy) \\
&= v_x^2 dx - v_y^2 dx - 2iv_x v_y dx + v_x^2 idy - v_y^2 idy + 2v_x v_y dy \\
&= v_x^2 dx - v_y^2 dx - 2iv_x^2 dy + v_x^2 idy - v_y^2 idy + 2v_y^2 dx \\
&= v_x^2 dx + v_y^2 dx - v_x^2 idy - v_y^2 idy \\
&= (v_x^2 + v_y^2) dx - (v_x^2 + v_y^2) idy \\
&= (v_x^2 + v_y^2)(dx - idy) \\
&= |v|^2\, d\bar{z}
\end{aligned} \tag{506}$$

Hence,

$$\bar{f} = i\,\frac{\rho}{2} \oint v^2\, dz = i\,\frac{\rho}{2} \oint \left(\frac{dw}{dz}\right)^2 dz \tag{507}$$

which is the statement of the Blasius lemma.

20.4 Kutta-Joukowski theorem

We now use Blasius' lemma to prove the Kutta-Joukowski lift theorem. For flow around a plane wing we can expand the complex potential in a *Laurent series*, and it must be of the form

$$\frac{dw}{dz} = a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \ldots \tag{508}$$

Higher powers of z cannot appear if the flow remains finite at $|z| \to \infty$ and, in this case, we can identify

$$a_0 = v_x(\infty) - iv_y(\infty). \tag{509}$$

In particular, if the wing moves along the x-axis and surrounding gas is at rest, then simply $a_0 = v_x(\infty)$.

To obtain the physical meaning of a_{-1}, we note that by virtue of the residues theorem[26]

$$a_{-1} = \frac{1}{2\pi i} \oint \frac{dw}{dz}\, dz \tag{510}$$

[26] Let's assume some otherwise analytic function $f(z)$ has a pole at $z = 0$. The residue is the coefficient a_{-1} of the Laurent series $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$. The residue theorem states that for a positively oriented simple closed curve γ around $z = 0$

$$\oint_\gamma f(z) dz = 2\pi i\; a_{-1}.$$

Computing the integral on the rhs. gives

$$
\begin{aligned}
\oint \frac{dw}{dz}\, dz &= \oint (v_x - iv_y)(dx + idy) \\
&= \oint (v_x dx + v_y dy) + i \oint (v_x dy - v_y dx)
\end{aligned} \tag{511}
$$

The last integral vanishes as the boundary is stream line, see Eq. (505), so that

$$
a_{-1} = \frac{1}{2\pi i} \oint \boldsymbol{v} \cdot d\boldsymbol{x} = \frac{\Gamma}{2\pi i}. \tag{512}
$$

where Γ is the circulation defined above.

To evaluate the rhs. in Eq. (507), we note that to leading order

$$
\left(\frac{dw}{dz}\right)^2 \simeq a_0 + 2\frac{a_0 a_{-1}}{z} = a_0 + \frac{a_0 \Gamma}{\pi i z} \tag{513}
$$

Thus, using the residue theorem, we find

$$
\bar{f} = f_x - i f_y = i\frac{\rho}{2}\left(2\pi i \frac{a_0 \Gamma}{\pi i}\right) = i\rho\Gamma a_0 = \rho\Gamma v_y(\infty) + i\rho\Gamma v_x(\infty). \tag{514}
$$

Recall that $F_D = \ell f_x$ and $F_L = \ell f_y$, this is indeed the generalization of our earlier results for drag and lift on a cylinder, if we identify $v_y(\infty) = 0$ and $v_x(\infty) = -u_0$. Note that the results $F_D = 0$ is again a manifestation of d'Alembert's paradox (now for arbitrarily shaped wings), which can be traced back to the fact that we neglected the viscosity terms in the Navier-Stokes equations.

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