18.366 Random Walks and Diffusion, Spring 2005, M. Z. Bazant.

Problem Set 1

Due at lecture on Th Feb 10.

1. **Rayleigh's walk.** Consider an isotropic random walk in three dimensions with IID displacements of length, a, given by the PDF,

$$p(\vec{x}) = \frac{\delta(r-a)}{4\pi a^2} \qquad (r = |\vec{x}|)$$

   (a) Derive the following formula for the PDF of the position after N steps,

$$P_N(\vec{x}) = \frac{1}{2\pi^2 r} \int_0^\infty u \sin(ur) \left[\frac{\sin(ua)}{ua}\right]^N du.$$

   (b) Using Laplace's method, derive the asymptotic formula,

$$P_N(\vec{x}) \sim \left(\frac{3}{2\pi a^2 N}\right)^{3/2} \exp\left(-\frac{3r^2}{2a^2 N}\right) \tag{1}$$

   as $N \to \infty$ for $r = O(\sqrt{N})$, consistent with the Central Limit Theorem.

   (c) *Extra Credit:* Derive the next term in the Gram-Charlier expansion of $P_N(\vec{x})$, and show that 'central region' where Eq. (1) holds is much wider, $r = O(N^{3/4})$.

2. **Cauchy's walk.** Consider a random walk in one dimension with independent, *non*-identical displacements given by Cauchy's PDF,

$$p_n(x) = \frac{A_n}{a_n^2 + x^2}$$

   for some positive sequence $\{a_n\}$.

   (a) Derive the characteristic function, $\hat{p}_n(k)$, and determine A_n. Explain why $\hat{p}_n(k)$ is not analytic at the origin.

   (b) Derive the PDF of the position after N steps, $P_N(x)$. For $a_n = a$ =constant, how does the half-width of $P_N(x)$ scale with N? Explain how your result is consistent with the Central Limit Theorem.

3. **Pearson's walk.** Write a computer program to simulate Pearson's random walk in the plane, where the steps have constant length $a = 1$ and uniformly distributed IID random angles. By simulating many long walks of N steps (e.g. $N = 1000$) starting from the origin, compute and plot the PDFs (normalized histograms) of A_N, the fraction of time steps, $1 \le n \le N$, when the walker is in the right half plane ($x > 0$) and of B_N, the fraction of time the walker is in the first quadrant ($x > 0, y > 0$). Clearly, the expected values are $\langle A_N \rangle = \frac{1}{2}$ and $\langle B_N \rangle = \frac{1}{4}$, but what are the most probable values? Plot several trajectories to illustrate your result.