Solutions to Problem Set 1

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1 Rayleigh's Random Walk

We consider an isotropic random walk in 3 dimensions with independent identical displacements of length a, given by the PDF

$$p(\vec{x}) = \frac{\delta(r-a)}{4\pi a^2} \qquad (r = |\vec{x}|)$$

for which we verify

$$\iiint p(\vec{x})\mathrm{d}\vec{x} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\delta(r-a)}{4\pi a^2} r^2 \sin\theta \,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = 1.$$

1.1 PDF of the position after n steps

The characteristic function is given by

$$\hat{p}(\vec{k}) = \left\langle e^{i\vec{k}\cdot\vec{x}} \right\rangle = \iiint p(\vec{x}) e^{i\vec{k}\cdot\vec{x}}\mathrm{d}\vec{x}.$$

If we choose the spherical coordinate system such that ϕ is the rotation angle around $\vec{k}$ (fixed in this integration), then $\vec{k}\cdot\vec{x} = kr\cos\theta$ and

$$\begin{aligned}
\hat{p}(\vec{k}) &= \frac{2\pi}{4\pi a^2}\int_0^{\pi}\int_0^{\infty} e^{ikr\cos\theta}\delta(r-a) r^2 \sin\theta \,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\pi} e^{ika\cos\theta} \sin\theta \,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{-1}^{1} e^{ikas}\mathrm{d}s \qquad\qquad (\text{with } s = \cos\theta) \\
&= \frac{\sin(ka)}{ka} \qquad (k = |\vec{k}|)
\end{aligned}$$

for which we verify again the normalization condition $\hat{p}(\vec{0}) = 1$. We know from the lecture that the PDF of the position after n steps is given by

$$P_n(\vec{x}) = \iiint \left[\hat{p}(\vec{k})\right]^n e^{-i\vec{k}\cdot\vec{x}} \frac{\mathrm{d}\vec{k}}{(2\pi)^3}.$$

*Based on solutions for problems 1 and 2 by Thierry Savin (2003), and for problem 3 by Chris H. Rycroft (2006).

Using the spherical coordinates system such that ϕ is now the rotation angle around $\vec{x}$ (fixed in this integration), we can write (again using $\vec{k}\cdot\vec{x} = kr\cos\theta$)

$$\begin{aligned} P_n(\vec{x}) &= \frac{1}{(2\pi)^2}\int_0^{\pi}\int_0^{\infty} e^{-ikr\cos\theta}\left[\frac{\sin(ka)}{ka}\right]^n k^2\sin\theta\,\mathrm{d}k\mathrm{d}\theta \\ &= \frac{1}{(2\pi)^2}\int_0^{\infty}\left[\frac{\sin(ka)}{ka}\right]^n k^2\left[\int_{-1}^{1} e^{-ikrs}\mathrm{d}s\right]\mathrm{d}k \qquad \text{with } s=\cos\theta. \end{aligned}$$

Therefore

$$\boxed{P_n(\vec{x}) = \frac{1}{2\pi^2 r}\int_0^{\infty} k\sin(kr)\left[\frac{\sin(ka)}{ka}\right]^n \mathrm{d}k.}$$

1.2 Asymptotic formula

We can write

$$P_n(\vec{x}) = \frac{1}{2\pi^2 r}\int_0^{\infty} k\sin(kr)e^{n\psi(k)}\mathrm{d}k$$

where

$$\psi(k) = \log\left[\frac{\sin(ka)}{ka}\right].$$

In the limit $n\to\infty$, we are interested in the region around $k=0$, where $\psi(k)$ is a maximum. Taylor expanding $\psi(k)$ at $k=0$ gives

$$\begin{aligned} \frac{\sin(ka)}{ka} &= 1 - \frac{(ka)^2}{3!} + \frac{(ka)^4}{5!} + O(k^6) \\ \psi(k) &= -\frac{(ka)^2}{6} - \frac{(ka)^4}{180} + O(k^6). \end{aligned}$$

For this part, we are just interested in the first term. We write

$$P_n(\vec{x}) \sim \frac{1}{2\pi^2 r}\int_0^{\infty} k\sin(kr)e^{-n(ka)^2/6}\mathrm{d}k$$

and using relation (1) shown in appendix A, we obtain

$$\boxed{P_n(\vec{x}) \sim \left(\frac{3}{2\pi a^2 n}\right)^{3/2}\exp\left(-\frac{3r^2}{2a^2 n}\right).}$$

1.3 Second Term

Taking into account the next term in the Taylor expansion of $\psi(k)$, and writing

$$e^{n\psi(\vec{k})} = e^{-n(ka)^2/6}e^{-n(ka)^4/180} = e^{-n(ka)^2/6} - \frac{n(ka)^4}{180}e^{-n(ka)^2/6}$$

we get

$$P_n(\vec{x}) \sim \frac{1}{2\pi^2 r}\left[\int_0^{\infty} k\sin(kr)e^{-n(ka)^2/6}\mathrm{d}k - \frac{na^4}{180}\int_0^{\infty} k^5\sin(kr)e^{-n(ka)^2/6}\mathrm{d}k\right].$$

After simplification, and using relations (1) and (2), we obtain

$$\boxed{P_n(\vec{x}) \sim \left(\frac{3}{2\pi a^2 n}\right)^{3/2} \exp\left(-\frac{3r^2}{2a^2 n}\right)\left[1 - \frac{3}{4n} + \frac{3}{2a^2}\frac{r^2}{n^2} - \frac{9}{20a^4}\frac{r^4}{n^3}\right].}$$

By introducing the scaling variable $\xi = r\sqrt{\frac{3}{2a^2 n}}$ we can write

$$\frac{a^3 P_n(\vec{x})}{(2\pi/3)^{-3/2}} \sim \frac{e^{-\xi^2}}{n^{3/2}}\left[1 - \frac{1}{n}\left(\frac{3}{4} - \xi^2 + \frac{\xi^4}{5}\right)\right]$$

and we see that the Central Limit Theorem holds as long as

$$\frac{1}{n}\left(\frac{3}{4} - \xi^2 + \frac{\xi^4}{5}\right) = O(1) \quad \Leftrightarrow \quad \xi^4 = O(n).$$

We conclude that the width of the central region is given by $r = O(n^{3/4})$.

2 Cauchy's Random Walk

We consider a random walk in one dimension with independent, nonidentical displacements, given by the PDF

$$p_n(x) = \frac{A_n}{x_n^2 + x^2}$$

where $x_n > 0$ for every n.

2.1 Characteristic function

The characteristic function for this PDF is given by

$$\hat{p}_n(k) = \left\langle e^{ikx}\right\rangle = \int_{-\infty}^{+\infty} p_n(x) e^{ikx}\mathrm{d}x = A_n \int_{-\infty}^{+\infty} \frac{e^{ikx}}{x_n^2 + x^2}\mathrm{d}x.$$

To evaluate this we consider the complex integral

$$\oint_{\mathcal{C}_R} \frac{e^{ikz}}{z^2 + x_n^2}\mathrm{d}z$$

in which the integrand exhibits 2 poles at $z = \pm i x_n$, and where $\mathcal{C}_R$ is one of the contours defined in figure 1, depending on the sign of k. For the first case where $k > 0$, the Residue Theorem gives us

$$\oint_{\mathcal{C}_R^+} \frac{e^{ikz}}{z^2 + x_n^2}\mathrm{d}z = 2i\pi \operatorname*{Res}_{z = i x_n}\left(\frac{e^{ikz}}{z^2 + x_n^2}\right).$$

The left-hand side can be expressed as

$$\oint_{\mathcal{C}_R^+} \frac{e^{ikz}}{z^2 + x_n^2}\mathrm{d}z = \int_{-R}^{+R} \frac{e^{ikx}}{x_n^2 + x^2}\mathrm{d}x + \int_{\Gamma_R^+} \frac{e^{ikz}}{z^2 + x_n^2}\mathrm{d}z$$

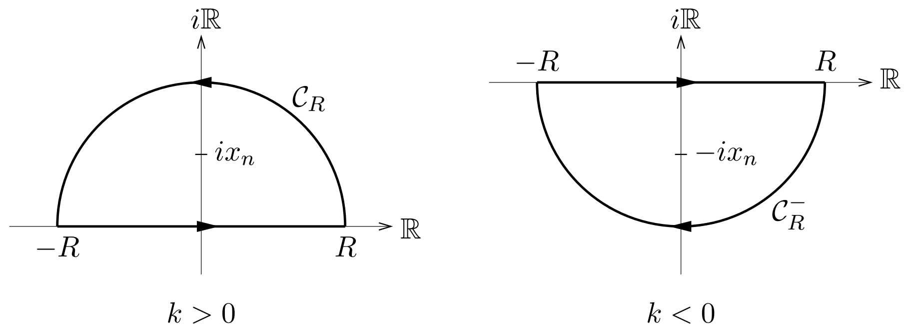

Figure 1: Contour $\mathcal{C}_R$. R is sufficiently big so that $\mathcal{C}_R$ runs around the pole.

where Γ_R^+ is the open contour over which z is imaginary (with $|z| = R$). On this contour, we have

$$\left| \int_{\Gamma_R^+} \frac{e^{ikz}}{z^2 + x_n^2} \mathrm{d}z \right| < \pi R \frac{e^{-kR}}{R^2 - x_n^2} \xrightarrow[R\to\infty]{} 0.$$

We conclude that when $R \to \infty$, we have

$$\oint_{\mathcal{C}_\infty^+} \frac{e^{ikz}}{z^2 + x_n^2} \mathrm{d}z = \int_{-\infty}^{+\infty} \frac{e^{ikx}}{x_n^2 + x^2} \mathrm{d}x.$$

The residue can be calculated as

$$\operatorname*{Res}_{z=ix_n} \left(\frac{e^{ikz}}{z^2 + x_n^2} \right) = \lim_{z\to ix_n} (z - ix_n) \frac{e^{ikz}}{z^2 + x_n^2} = \frac{e^{-kx_n}}{2ix_n}$$

and hence we obtain

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{x_n^2 + x^2} \mathrm{d}x = \frac{\pi}{x_n} e^{-kx_n} \qquad \text{for } k > 0.$$

Using the other contour for $k < 0$, we show similarly that

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{x_n^2 + x^2} \mathrm{d}x = \frac{\pi}{x_n} e^{kx_n} \qquad \text{for } k < 0.$$

Thus we can write[1]

$$\hat{p}_n(k) = \frac{\pi A_n}{x_n} e^{-|k|x_n};$$

the normalization condition gives us

$$\hat{p}_n(0) = 1 \quad \Leftrightarrow \quad A_n = \frac{x_n}{\pi}$$

and therefore

$$\boxed{\hat{p}_n(k) = e^{-x_n|k|}.}$$

We see that $\hat{p}_n(k)$ is continuous at $k = 0$, but $\hat{p}_n'(k)$ is not, since it doesn't have the same value at $k = 0^+$ and $k = 0^-$. In terms of $p(x)$, this is because $\langle x \rangle_n = -i\,\hat{p}_n'(k = 0)$ is not defined.

[1] Another rigorous way to prove this formula is to calculate the integral

$$\int_{-\infty}^{+\infty} e^{-ikx} e^{-|k|x_n} \frac{\mathrm{d}k}{2\pi} = \frac{1}{\pi} \frac{x_n}{x_n^2 + x^2}$$

as it is shown in paragraph 2.2, and then use the Inverse Fourier Transform Theorem.

2.2 PDF of the position after n steps

From class, we know

$$
\begin{aligned}
P_n(x) &= \int_{-\infty}^{+\infty} e^{-ikx}\left[\prod_{j=1}^{n}\hat{p}_j(k)\right]\frac{\mathrm{d}k}{2\pi} \\
&= \int_{-\infty}^{+\infty} e^{-ikx}e^{-|k|X_n}\frac{\mathrm{d}k}{2\pi} \qquad \text{where} \quad X_n = \sum_{j=1}^{n} x_j \\
&= \int_{0}^{+\infty} e^{-(ix+X_n)k}\frac{\mathrm{d}k}{2\pi} + \int_{-\infty}^{0} e^{-(ix-X_n)k}\frac{\mathrm{d}k}{2\pi} \\
&= \frac{1}{2\pi}\left(\frac{1}{ix+X_n} - \frac{1}{ix-X_n}\right) \qquad \text{since } X_n > 0.
\end{aligned}
$$

Thus we obtain

$$\boxed{P_n(x) = \frac{1}{\pi}\frac{X_n}{X_n^2+x^2}} \qquad \text{with} \quad X_n = \sum_{j=1}^{n} x_j.$$

As noticed earlier, the variance of the PDF is infinite. Since one of the assumptions of the CLT was violated, we can not apply the CLT to this problem and the resulting PDF is not in the form of the gaussian distribution.

For the case of identical steps $x_n = a$ we get

$$p(x) = \frac{a}{\pi}\frac{1}{a^2+x^2} \qquad \text{and} \qquad P_n(x) = \frac{na}{\pi}\frac{1}{n^2a^2+x^2}.$$

It is not surprising to not observe a gaussian behavior for the PDF after n steps. That is, the scaling $\xi = \frac{x}{a\sqrt{n}}$ is not appropriate. However, by scaling $\xi = \frac{x}{na}$, the normalized PDF for the variable ξ is written

$$\tilde{P}_n(\xi) = naP_n(x) = \frac{1}{\pi}\frac{1}{1+\xi^2} \qquad \text{independent of } n.$$

3 Ergodicity breaking

3.1 Generating the Cauchy random walk

Let Y be a random variable which is uniformly distributed on the range $[-\pi/2, \pi/2]$. The cumulative density function is

$$\mathbb{P}(Y < y) = \frac{2y+\pi}{2\pi}$$

If $X = \tan Y$, then

$$
\begin{aligned}
\mathbb{P}(\tan Y < \tan y) &= \frac{2y+\pi}{2\pi} \\
\mathbb{P}(X < \tan y) &= \frac{2y+\pi}{2\pi} \\
\mathbb{P}(X < x) &= \frac{2\tan^{-1}x+\pi}{2\pi},
\end{aligned}
$$

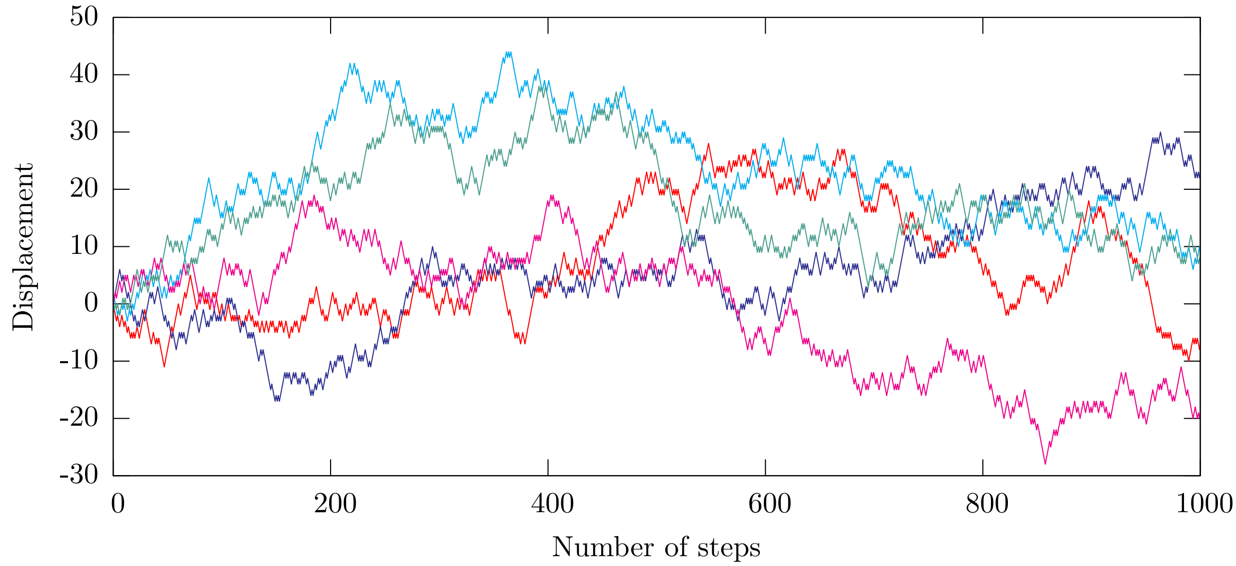

Figure 2: Five Bernoulli random walks.

which is the cumulative density function of X. By differentiating this, the PDF of X is given by

$$\begin{aligned} p(x) &= \frac{d}{dx}\left(\frac{2\tan^{-1}x+\pi}{2\pi}\right) \\ &= \frac{1}{\pi(1+x^2)}. \end{aligned}$$

Thus X is distributed according the Cauchy distribution; this gives us simple method for generating a Cauchy random walk, by first generating random numbers uniformly distributed on $[-\pi/2, \pi/2]$ and then taking the tangent. Figures 2 and 3 show five sample Bernoulli and Cauchy random walks respectively. It is clear the two walks have very different structures, with the Cauchy walkers exhibiting very large single-step jumps.

3.2 Finding the distribution of α_N

Appendix B contains a C++ code for calculating the steps of a Bernoulli or Cauchy random walk of length N, and computing the proportion of the steps α_N which satisfy $x > 0$.

The code was run for 4×10^7 walks of length $N = 10^5$, and the resulting distributions of α_N are shown in figure 4. The curves for the Bernoulli and Cauchy random walks appear indistinguishable.

3.3 The limiting distribution

The two curves for α_N very closely fit the functional form $(x - x^2)^{-1/2}/\pi$.

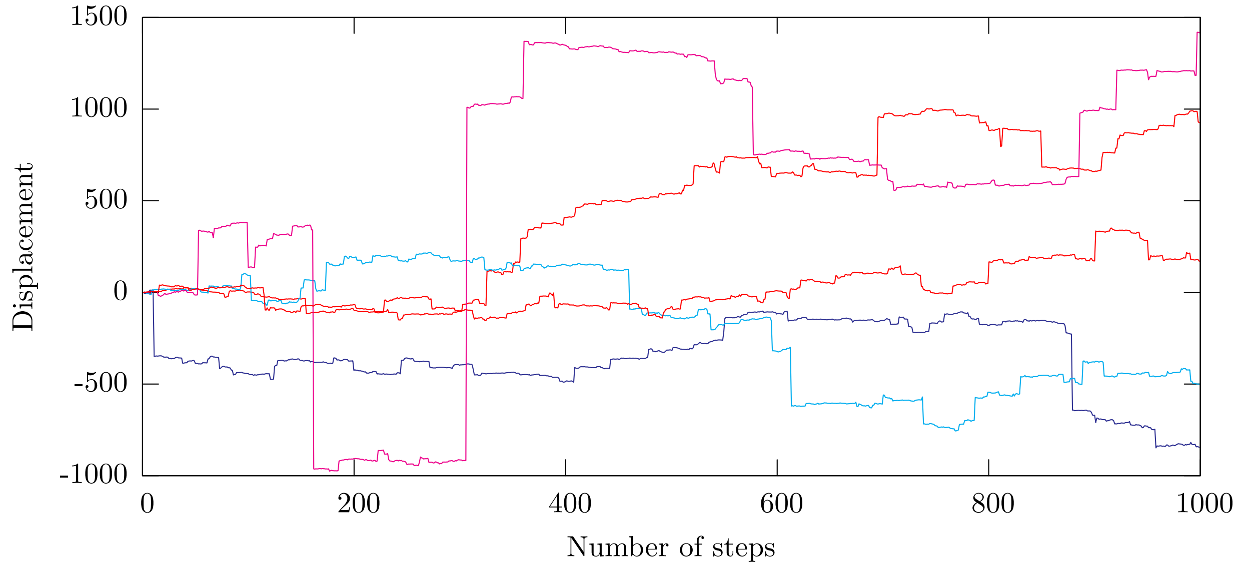

Figure 3: Five Cauchy random walks.

A Appendix

We consider the integral

$$\begin{aligned}\int_0^\infty \cos(k\alpha)e^{-\beta k^2}\mathrm{d}k &= \frac{1}{2}\int_{-\infty}^\infty \cos(k\alpha)e^{-\beta k^2}\mathrm{d}k \\ &= \frac{1}{4}\left[\int_{-\infty}^\infty e^{-ik\alpha-\beta k^2}\mathrm{d}k + \int_{-\infty}^\infty e^{ik\alpha-\beta k^2}\mathrm{d}k\right] \\ &= \frac{e^{-\frac{\alpha^2}{4\beta}}}{4}\left[\int_{-\infty}^\infty e^{-\beta\left(k+i\frac{\alpha}{2\beta}\right)^2}\mathrm{d}k + \int_{-\infty}^\infty e^{-\beta\left(k-i\frac{\alpha}{2\beta}\right)^2}\mathrm{d}k\right].\end{aligned}$$

To calculate the first integral, let us now consider the complex integral

$$\oint_{\mathcal{C}_L} e^{-\beta z^2}\mathrm{d}z$$

where $\mathcal{C}_L$ is the contour defined in figure 5.

Cauchy's Theorem implies that this integral vanishes. Expanding the integration on each side of the contour $\mathcal{C}_L$ gives

$$\begin{aligned}\oint_{\mathcal{C}_L} e^{-\beta z^2}\mathrm{d}z &= \int_{-L}^{L} e^{-\beta x^2}\mathrm{d}x + \int_0^{\frac{\alpha}{2\beta}} e^{-\beta(L+iy)^2}\mathrm{d}y \\ &\quad + \int_L^{-L} e^{-\beta\left(x+i\frac{\alpha}{2\beta}\right)^2}\mathrm{d}x + \int_{\frac{\alpha}{2\beta}}^0 e^{-\beta(-L+iy)^2}\mathrm{d}y \\ &= 0\end{aligned}$$

We see immediately to see that the second and fourth term in this expansion vanish as $L \to \infty$. In

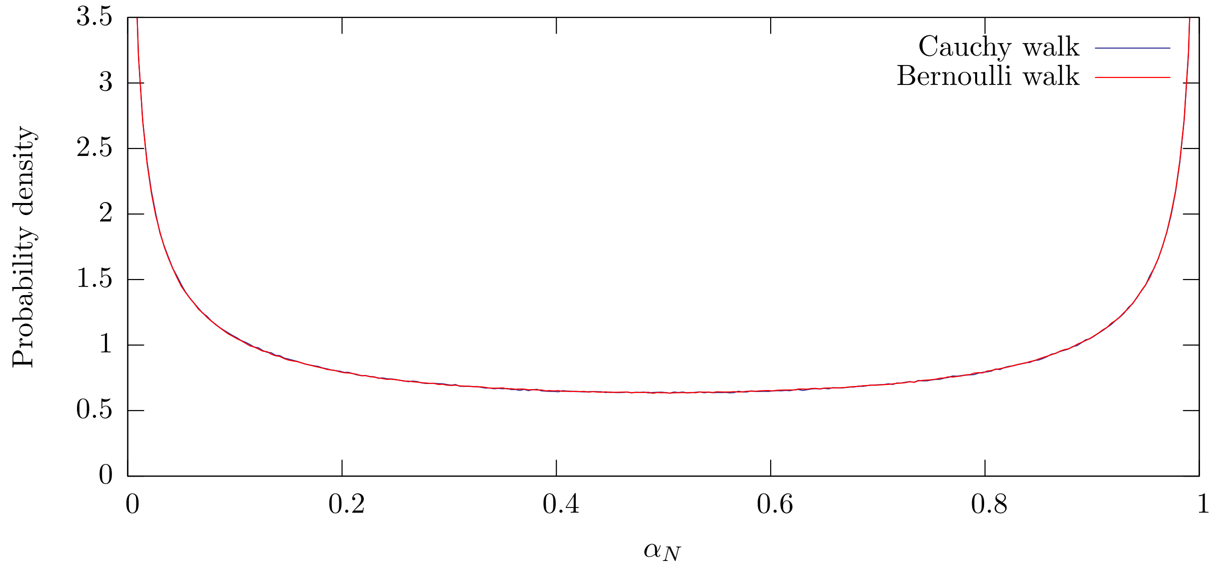

Figure 4: Computed probability distributions of α_N for the Bernoulli and Cauchy random walks.

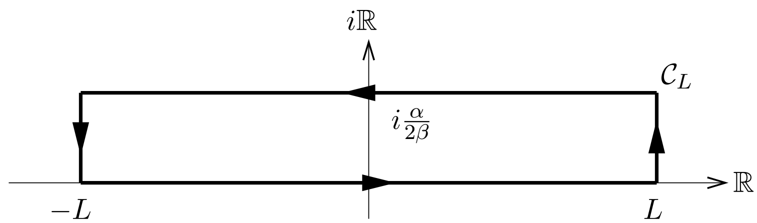

Figure 5: Contour $\mathcal{C}_L$.

this limit, we can conclude

$$\int_{-\infty}^{\infty} e^{-\beta\left(x+i\frac{\alpha}{2\beta}\right)^2} \mathrm{d}x = \int_{-\infty}^{\infty} e^{-\beta x^2} \mathrm{d}x = \sqrt{\frac{\pi}{\beta}}.$$

Similarly, we can also prove

$$\int_{-\infty}^{\infty} e^{-\beta\left(x-i\frac{\alpha}{2\beta}\right)^2} \mathrm{d}x = \int_{-\infty}^{\infty} e^{-\beta x^2} \mathrm{d}x = \sqrt{\frac{\pi}{\beta}}.$$

Thus we obtain

$$\int_0^{\infty} \cos(k\alpha) e^{-\beta k^2} \mathrm{d}k = \frac{\sqrt{\pi}}{2\sqrt{\beta}} e^{-\frac{\alpha^2}{4\beta}}.$$

Using the equalities

$$\int_0^{\infty} k \sin(k\alpha) e^{-\beta k^2} \mathrm{d}k = -\frac{d}{d\alpha} \int_0^{\infty} \cos(k\alpha) e^{-\beta k^2} \mathrm{d}k$$
$$\int_0^{\infty} k^5 \sin(k\alpha) e^{-\beta k^2} \mathrm{d}k = -\frac{d^5}{d\alpha^5} \int_0^{\infty} \cos(k\alpha) e^{-\beta k^2} \mathrm{d}k$$

we obtain the useful relations

$$\int_0^\infty k\sin(k\alpha)e^{-\beta k^2}\mathrm{d}k = \frac{\alpha\sqrt{\pi}}{4\beta^{3/2}}e^{-\frac{\alpha^2}{4\beta}}, \tag{1}$$

$$\int_0^\infty k^5\sin(k\alpha)e^{-\beta k^2}\mathrm{d}k = \frac{\alpha\sqrt{\pi}}{64\beta^{11/2}}e^{-\frac{\alpha^2}{4\beta}}\left(\alpha^4 - 20\alpha^2\beta + 60\beta^2\right). \tag{2}$$

B C++ code for finding the distribution of α_N

The code below will calculate the distribution of α_N for the Cauchy random walk. To generate the distribution of α_N for the Bernoulli walk, uncomment the lines labeled `Bernoulli` and comment the lines labeled `Cauchy`.

```
#include <string>
#include <iostream>
#include <cstdio>
#include <cmath>
using namespace std;

const double p=3.14159265358979323846264338327950;
const long n=100000;  //Number of steps in a walk
const long w=40000000;  //Number of walkers

int main () {
        long i,j,c,a[n+1];double y;
        for(i=0;i<=n;i++) a[i]=0;
//      long x;             // Bernoulli
        double x;           // Cauchy
        for(j=0;j<w;j++) {
                x=0;c=0;
                for(i=0;i<n;i++) {
//                      x+=rand()%2==0?-1:1;  // Bernoulli
                        x+=tan(((double(rand())+0.5)
                           /RAND_MAX-0.5)*p); // Cauchy
                        if (x>0) c++;
                }
                a[c]++;
        }
        for(i=0;i<=n;i++) {
                y=double(i)/n;
                cout << i << " " << y << " " << a[i] << endl;
        }
}
```