Problem Set Number 2, 18.385j/2.036j
MIT (Fall 2014)

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1 Inverse function problem #01.

Statement: Inverse function problem #01.

Consider the following equation

$$y = x + \sin(x) = f(x), \tag{1.0.1}$$

where, in particular, $f(0) = 0$ and $f'(0) = 2 \neq 0$. The inverse function theorem guarantees that: there is a neighborhood of $x = 0$ where f has a unique inverse, $x = X(y)$, such that $X(0) = 0$. Furthermore, since f is an analytic function, X is an analytic function. This means that X has a Taylor series

$$X = \sum_{n=0}^{\infty} x_n\, y^n, \tag{1.0.2}$$

which converges for $|\lambda|$ small enough. **Find x_1, x_3, x_5, and x_n for all even n.**

2 Find and classify bifurcations problem #01.

Statement: Find and classify bifurcations problem #01.

For equation (2.0.1) below, find the values of r at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

$$\frac{dx}{dt} = r - \frac{x^2}{1+x^2}. \tag{2.0.1}$$

3 Irreversible switch using a saddle node and a transcritical bifurcation.

Statement: Irreversible switch using a saddle node and a transcritical bifurcation.

Imagine a system[1] with a controlling parameter r, and with (at most) two distinct stable equilibrium states: $x_1 = x_1(r)$ and $x_2 = x_2(r)$. In particular, such that *infinity is unstable* — that is: for every solution $x = x(t)$ there exists a constant $M > 0$ such that $|x| < M$ for t large enough. Furthermore:

A. There is a value $r = r_s =$ **switch value** such that: for $r > r_s$ both states exist and are stable — so that the system can be in either one of them.

B. For $r < r_s$ only the state x_1 exists and it is stable.

C. Both $x_1(r)$ and $x_2(r)$ are continuous functions of r (though, maybe, not smooth), and $|x_1(r) - x_2(r)|$ is bounded away from zero.

Such a system, if started in the state x_2 for $r > r_s$, remains in x_2 for as long as r varies (slowly enough) in the range $r > r_s$. Once r crosses below the threshold r_s, the system switches to x_1, and remains there for all values of r. A switch back to x_2 is not produced by slow variations in r. The condition in item **C** is important, for otherwise small perturbations could produce an "accidental" switch if x_1 and x_2 get very close.

Remark 3.0.1 *A "standard" (reversible) switch [e.g.: a thermostat], operates using hysteresis. For such systems there are two switching values $r_1 < r_2$, with only x_2 stable for $r > r_2$, only x_1 stable for $r < r_1$, and both states stable for $r_1 \le r \le r_2$. Then the system jumps from x_2 to x_1 as r is lowered below r_1, and back to x_2 as r is raised above r_2.* ♣

Construct an irreversible switch, using a 1-D system of the form

$$\frac{dx}{dt} = f(x,\, r), \tag{3.0.1}$$

[1] A "switch".

with the behavior caused by two bifurcations: a trans-critical and a saddle node *(no other bifurcations should occur!)* **Then draw the bifurcation diagram.**

Hint: It is very easy to construct an explicit example in which f in equation (3.0.1) is a cubic polynomial in x, and it is linear in the parameter r.

Remark 3.0.2 (Switch uniqueness). *Even for a 1-D system such as the one in (3.0.1), there is an infinite number of possible bifurcation diagrams that yield a switch, with various types of bifurcations involved.*[2] *However, if the restriction that there should be only two bifurcations (one saddle-node and one transcritical) is imposed, then there are only two possible topologies for the switch bifurcation diagram. This problem asks you to produce an example of one such switch.* ♣

4 Toy model for shell buckling.

Statement: Toy model for shell buckling.

Hold a ping-pong ball between your thumb and index fingers and squeeze it. If you do not apply enough force, the ball will deform slightly with a purely elastic response. But, if you push hard enough, the ball will buckle and you will make a (permanent) dent on it — and the ball will be ruined. This is the phenomena of (thin) shell buckling.

Shell buckling is a very rich phenomena,[3] way beyond the scope of this course. Here we will study an extremely simplified (1-D) version of this phenomena (the emphasis here being on "toy" model) where all the geometrical richness of the original setting is gone, and only the buckling bifurcation remains.

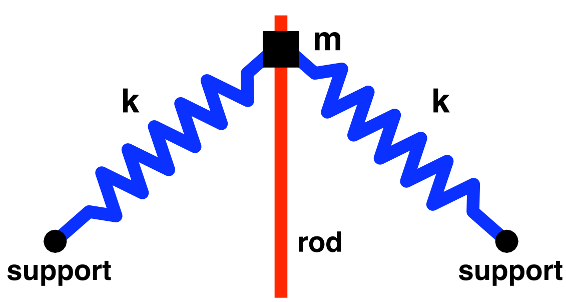

Figure 0.1: Toy model for shell buckling. A bead of mass m (black square) can slide along a rigid vertical rod (in red). The bead is connected by two equal springs (in blue), with spring constant k, to two supports placed symmetrically on each side of the rod. See the text for further details.

A sketch depicting the model is shown in figure 0.1. Further assumptions and notation are:

[2] This is the subject of another problem: "Irreversible switches; classification."

[3] Lots of interesting and important questions arise. For example: What is the shape of the dent that forms? The dent's edges have sharp corners: why these corners form, and how do they propagate as further pressure is applied?

1. Idealize the bead as a point mass.

2. Let x be the vertical distance, along the rod, of the bead from the horizontal line joining the spring supports. Let $x > 0$ if the bead is above the supports and $x < 0$ if below.

3. Let $\boldsymbol{h > 0}$ be the distance of the spring supports from the rod, and let $\boldsymbol{L > 0}$ be the springs equilibrium length. **Assume $\boldsymbol{L > h}$**, so that the springs are under compression for $x = 0$.

4. Hook's law applies to the springs. Thus they exert a force of magnitude $F = k(\ell - L)$, where ℓ is the spring length, along the spring axis, pushing if $\ell < L$, and pulling if $\ell > L$.

5. When the bead slides along the rod, the motion is opposed by a friction force of magnitude $b\,\dot{x}$, where $b > 0$ is a constant.

6. Because the rod is rigid, we need to consider only the vertical components of the various forces that act on the bead. These forces are: (i) Gravity, of magnitude $m\,g$, pointing down. (ii) The forces by the springs. (iii) Friction along the rod.

PROBLEM TASKS:

A. *Derive an ode for the bead position, and write it in appropriate a-dimensional variables.*[4]

B. *Assume that friction is large, so that inertia can be neglected. Exactly which a-dimensional number has to be small for friction to be "large"?*

C. *Analyze the bifurcations that occur for the equation resulting from item B, as the bead mass changes — in this toy model, increasing the bead mass plays the role of squeezing harder on the ping-pong ball. What type of bifurcation(s) occur?*

Hint: It is a bad idea to try to do this by attempting to solve for the critical points and bifurcation thresholds analytically. A qualitative, graphical, analysis is the best way to go.

D. *The picture in figure* 0.1 *corresponds, in this toy model, to the ping-pong ball in a more-or-less spherical shape. What is the "buckled" state?*

E. *What a-dimensional parameter controls when bifurcations happen?*

$$\textbf{Assume that the ratio } \boldsymbol{\gamma = L/h > 1} \textbf{ is kept fixed.} \tag{4.0.1}$$

5 Bifurcations in the circle problem #04.

Statement: Bifurcations in the circle problem #04.

For equation (5.0.1) find the values of r at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram for the

[4] Suggestion: to a-dimensionalize use h for length and $b/(2\,k)$ for time.

fixed points versus r, including the flow direction and the stability of the various branches of solutions (solid lines for stable branches and dashed ones for unstable ones).

$$\frac{d\theta}{dt} = (r - \sin(\theta))\,\sin(\theta), \tag{5.0.1}$$

where θ is an angle (in radians). Note that *the bifurcation diagram — which is periodic in θ — should be for a 2π range in θ, and a range of r that includes all the bifurcations.*

6 Problem 03.04.08 - Strogatz (Find and classify bifurcations).

Statement for problem 03.04.08.

For the following equation, find the values of r at which bifurcations occur, and classify those as saddle node, transcritical or pitchfork (supercritical or subcritical). Finally, sketch the bifurcation diagram of fixed points, x^* versus r.

$$\frac{dx}{dt} = r\,x - \frac{x}{1+x^2}. \tag{6.0.1}$$

Extra question: Notice that something "strange" happens for $r = 0$ in the bifurcation diagram. Is this a bifurcation? If so, which type? Does the "principle of conservation of stability" apply? *Hint: look at the equation satisfied by $y = 1/x$.*

7 Problem 03.04.11 - Strogatz (An interesting bifurcation diagram).

Statement for problem 03.04.11.

(An interesting bifurcation diagram). Consider the system

$$\frac{dx}{dt} = rx - \sin(x)\,. \tag{7.0.1}$$

A. For the case $r = 0$, find and classify the fixed points, and sketch the vector field.

B. Show that, when $r > 1$, there is only one fixed point. What kind of fixed point is it?

C. As r decreases from ∞ to 0, classify **all** the bifurcations that occur.

D. For $0 < r \ll 1$, find an approximate formula for the values of r at which bifurcations occur.

E. Now classify all the bifurcations that occur as r decreases from 0 to $-\infty$.

F. Plot the bifurcation diagram for $-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

8 Problem 03.04.14 - Strogatz (Subcritical Pitchfork).

Statement for problem 03.04.14.

(Subcritical Pitchfork). Consider the system

$$\frac{dx}{dt} = rx + x^3 - x^5\,, \tag{8.0.1}$$

which exhibits a subcritical pitchfork bifurcation.

a) Find algebraic expressions for all the fixed points as r varies.

b) Sketch the vector fields as r varies. Be sure to indicate all the fixed points and their stability.

c) Calculate $\boldsymbol{r_c}$, the parameter value at which the nonzero fixed points are born in a saddle-node bifurcation.

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9 Problem 03.06.06 - Strogatz (Patterns in fluids).

Statement for problem 03.06.06.

(Patterns in fluids). G. Ahlers (1989)[5] gives a fascinating review of experiments on one-dimensional patterns in fluid systems. In many cases, the patterns first emerge via supercritical or subcritical pitchfork bifurcations from a spatially uniform state. Near the bifurcation, the dynamics of the amplitude of the patterns are given approximately by

$$\tau\frac{dA}{dt} = \epsilon A - gA^3 \quad \text{in the supercritical case,} \tag{9.0.1}$$

or

$$\tau\frac{dA}{dt} = \epsilon A - gA^3 - kA^5 \quad \text{in the subcritical case.} \tag{9.0.2}$$

Here $A = A(t)$ is the amplitude of the pattern, $\tau > 0$ is a typical time scale, and ϵ is a small dimensionless parameter that measures the distance from the bifurcation. The parameter g **is positive in the supercritical case, whereas** $g < 0$ **and** $k > 0$ **in the subcritical case.** (*In this context, the equation* $\tau\dot{A} = \epsilon A - gA^3$ *is often called the* **Landau equation.**)

a) Dubois and Bergé (1978)[6] studied the supercritical bifurcation that arises in Rayleigh–Bénard convection, and showed experimentally that the steady state amplitude depends on ϵ according to the power law $A^* \propto \epsilon^\beta$, where $\beta = 0.50 \pm 0.01$. What does the Landau equation predict?

[5] Ahlers, G. (1989) Experiments on bifurcations and one-dimensional patterns in nonlinear systems far from equilibrium. In D. L. Stein, ed. *Lectures in the Science of Complexity* (Addison-Wesley, Reading, MA).

[6] Dubois, M., and Bergé, P. (1978) Experimental study of the velocity field in Rayleigh–Bénard convection. *J. Fluid Mech.* **85**, 641.

b) The equation $\tau\dot{A} = \epsilon A - gA^3 - kA^5$ is said to undergo a **transcritical**[7] **bifurcation** when $g = 0$; this case is the borderline between supercritical and subcritical bifurcation. Find the relation between A^* and ϵ when $g = 0$.

c) In experiments on Taylor–Couette vortex flow, Aitta et al. (1985)[8] were able to change the parameter g continuously from positive to negative by varying the aspect ratio of their experimental set-up. Assuming that the equation is modified to

$$\frac{dA}{dt} = h + \epsilon A - gA^3 - kA^5\,, \tag{9.0.3}$$

where $h > 0$ is a slight imperfection, sketch the bifurcation diagram of A^* versus ϵ in the three cases: $g > 0$, $g = 0$, and $g < 0$. Then look up at the actual data in Aitta et al. (1985, figure 2) or see Ahlers (1989, figure 15).

d) In the experiments of part (c), the amplitude $A(t)$ was found to evolve toward a steady state in the manner shown in figure 2 of the book (page 88) — redrawn from Ahlers (1989), figure 18. The results are for the imperfect subcritical case $g < 0$, $h \neq 0$. In the experiments, the parameter ϵ was switched at $t = 0$ from a negative value to a positive value ϵ_f (in the figure ϵ_f increases from the bottom to the top.)

Explain intuitively why the curves have this strange shape. Why do the curves for large ϵ_f go almost straight up to their steady state, whereas the curves for small ϵ_f rise to a plateau before increasing sharply to their final level? (**Hint:** Graph $\dot{A}$ versus A for different ϵ_f.)

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10 Problem 04.01.01 - Strogatz (Define a flow in circle).

Statement for problem 04.01.01.

For which real values of a does the equation

$$\frac{d\theta}{dt} = \sin(a\theta)$$

give a well defined vector field on the circle?

THE END.

[7] **WARNING:** This is a rather unfortunate choice of name! **Do not to confuse** this situation with the **transcritical bifurcation** introduced in section 3.2 of the book. **They are not the same thing!**

[8] Aitta, A., Ahlers, G., and Cannell, D. S. (1985) Transcritical phenomena in rotating Taylor–Couette flow. *Phys. Rev. Lett.* **54**, 673.